Math 270B: Numerical Analysis (Part B)
Winter quarter 2023

Homework Assignment 4

Due: 10:00 am, Wednesday, February 8, 2023.

1. Let $f \in C([a, b])$ and define

$$\mu_n(f) = \int_a^b x^n f(x)\,dx, \qquad n = 0, 1, \ldots$$

Show that $f(x) = 0$ for all $x \in [a, b]$ if and only if $\mu_n(f) = 0$ for all $n = 0, 1, \ldots$

2. Let $n \geq 0$ be an integer and T_n the n-th Chebyshev polynomial. Show that

$$\int_{-1}^{1} [T_n(x)]^2 = 1 - \frac{1}{4n^2 - 1}.$$

3. Let $n \geq 0$ be an integer and T_n the nth Chebyshev polynomial of first kind. Let $P \in \mathcal{P}_n$ satisfy that $|P(x)| \leq 1$ for all $x \in [-1, 1]$. Show that

$$|P(y)| \leq |T_n(y)| \qquad \forall y \notin [-1, 1].$$

4. Define $\chi(x) = -1$ if $-1 \leq x < 0$ and $\chi(x) = 1$ if $0 \leq x \leq 1$.
   1. Show that $\inf_{f \in C([-1,1])} \sup_{-1 \leq x \leq 1} |f(x) - \chi(x)| = 1$, and that there exist infinitely many $f \in C([-1, 1])$ such that $\sup_{-1 \leq x \leq 1} |f(x) - \chi(x)| = 1$.
   2. Show that

   $$\inf_{f \in C([-1,1])} \int_{-1}^{1} |f(x) - \chi(x)|^2 dx = 0,$$

   and that there exists no $f \in C([-1, 1])$ such that

   $$\int_{-1}^{1} |f(x) - \chi(x)|^2 dx = 0.$$

5. Let $a, b \in \mathbb{R}$ with $a < b$, $f \in C([a, b])$, and $\varepsilon > 0$. Show that there exists a polynomial p such that $\|f - p\|_{L^2(a,b)} < \varepsilon$.

6. Find the least-squares approximation of $f(x) = x^4$ in $\mathcal{P}_1$ over $[0, 1]$.

7. Let $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathcal{P}_n$ be the least-squares approximation of a given $f \in C([0, 1])$ over $[0, 1]$. Find the coefficient matrix of the linear system that determines $a_0, \ldots, a_n$.

8. Let

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(x^2 - 1 \right)^n \right], \qquad n = 0, 1, \ldots$$

be the Legendre polynomials. Let $n \geq 1$. Prove directly by Rolle's Theorem that P_n has n simple roots in $(-1, 1)$.