Math 270B: Numerical Analysis (Part B)
Winter quarter 2023

Homework Assignment 6

Due: 10:00 am, Friday, February 24, 2023.

1. Find the polynomial $p \in \mathcal{P}_3$ of the form $p(x) = c_0 + c_1 x + c_3 x^3$ that interpolates a given function $f \in C([0,3])$ at $x = 0, 2, 3$.

2. Let $x_0 = 2$, $x_1 = 3$, $x_2 = 5$, $x_3 = 6$ and $y_0 = 5$, $y_1 = 2$, $y_2 = 3$, $y_3 = 4$. Let $p \in \mathcal{P}_3$ be the unique polynomial that interpolates y_j at x_j $(j = 0, 1, 2, 3)$. Calculate p by using: (1) Lagrange's formula; and (2) Newton's formula.

3. Let $f(x) = x^4 - x^2 + 17x + 1$. Let $p \in \mathcal{P}_{20}$ interpolates f at $x_j = 2^j$ $(j = 0, \dots, 20)$. Compute $p(0)$.

4. Let $f(x) = x^4 - x^2 + 17x + 1$ and $x_k = k$ $(k = 0, 1, \dots, 20)$. Calculate $f[x_0, \dots, x_4]$ and $f[x_0, \dots, x_{20}]$.

5. Let $x_0, \dots, x_n$ be $n + 1$ distinct real numbers. Let $l_j(x)$ be the associated Lagrange basis polynomials. Show that $\sum_{j=0}^{n} (x - x_j)^k \, l_j(x) = 0$ for all $k = 1, \dots, n$.

6. Recall for $n \geq 1$ that the Chebyshev polynomial $T_n(x)$ has n distinct roots $x_j = \cos\theta_j$ with $\theta_j = (2j-1)\pi/2n$ $(j = 1, \dots, n)$. Denote by $L_{n-1} : C([-1,1]) \to \mathcal{P}_{n-1}$ the associated Lagrange interpolation operator. Show that
$$(L_{n-1}f)(x) = \frac{1}{n} \sum_{j=1}^{n} f(x_j) \frac{(-1)^{j-1} \sin\theta_j T_n(x)}{x - x_j} \qquad \forall f \in C([-1,1]).$$

7. Let $f \in C([a,b])$. Show that, for each integer $n \geq 1$, there exist n distinct points $x_1^{(n)}, \dots, x_n^{(n)}$ such that
$$\|f - L_{n-1}f\|_{C([a,b])} \to 0 \qquad \text{as } n \to \infty,$$
where $L_{n-1}f \in \mathcal{P}_{n-1}$ is the Lagrange interpolation polynomial of f at $x_1^{(n)}, \dots, x_n^{(n)}$.

8. Let $Q_n \in \mathcal{P}_n$ $(n = 0, 1, \dots)$ be orthonormal polynomials on $[0,1]$. Fix $n \geq 2$. Let $x_1, \dots, x_n$ be the n distinct roots of $Q_n(x)$ in $(0,1)$, and $l_1, \dots, l_n$ be the associated Lagrange basis polynomials. Prove that $l_1, \dots, l_n$ are orthogonal on $[0,1]$ and that
$$\sum_{j=1}^{n} \int_0^1 [l_j(x)]^2 \, dx = 1.$$

9. The ReLU (Rectified Linear Unit) activation function used in neural networks is defined by $\mathrm{ReLU}(x) = \max(x, 0)$ (also denoted by x^+ or x_+) for any $x \in \mathbb{R}$. Prove for any integer $k \geq 0$ and any $x \in \mathbb{R}$ with at least one of them nonzero that
$$[\mathrm{ReLU}(x)]^k + (-1)^k [\mathrm{ReLU}(-x)]^k = x^k.$$