

Math 270B: Numerical Analysis (Part B)
Winter quarter 2023

Homework Assignment 7

Due: 10:00 am, Friday, March 3, 2023.

1. Let $x_0, \dots, x_n \in [a, b]$ be distinct, $L_n : C([a, b]) \rightarrow \mathcal{P}_n$ the corresponding Lagrange interpolation operator, and l_0, \dots, l_n the associated Lagrange basis polynomials. Prove the following:

- (1) $\|L_n f\|_{C([a, b])} \leq \lambda_n \|f\|_{C([a, b])}$ for any $f \in C([a, b])$, where $\lambda_n = \max_{a \leq x \leq b} \sum_{j=0}^n |l_j(x)|$;
- (2) There exists a $f_0 \in C([a, b])$ such that $\|f_0\|_{C([a, b])} = 1$ and $\|L_n f_0\|_{C([a, b])} = \lambda_n \|f_0\|_{C([a, b])}$.

2. Let $n \geq 0$ be an integer and X_n the set of $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ such that x_0, \dots, x_n are distinct points in $[-1, 1]$. Prove that the function $N(x_0, \dots, x_n) = \max_{-1 \leq x \leq 1} |(x - x_0) \dots (x - x_n)|$ has a unique minimizer in X_n . Find this minimizer and the minimum value.

3. Let $n \geq 1$ and Q_0, \dots, Q_n be orthogonal polynomials on $[a, b]$. Let x_1, \dots, x_n be the n distinct zeros of Q_n in $[a, b]$. Prove for any distinct points $y_1, \dots, y_n \in [a, b]$ that

$$\int_a^b (x - x_1)^2 \dots (x - x_n)^2 dx \leq \int_a^b (x - y_1)^2 \dots (x - y_n)^2 dx.$$

4. Let Q_0, Q_1, \dots be orthogonal polynomials on $[0, 1]$. For each $n \geq 1$, let $x_1^{(n)}, \dots, x_n^{(n)}$ denote the n distinct zeros of Q_n in $[0, 1]$ and $L_{n-1} : C([0, 1]) \rightarrow \mathcal{P}_{n-1}$ the associated Lagrange interpolation operator. Let $f \in C([0, 1])$. Prove that $\|f - L_{n-1}f\|_{L^2(0,1)} \rightarrow 0$.

5. Let $N \geq 1$ be an integer, $h = (b - a)/N$, and $x_j = a + jh$, $j = 0, \dots, N$. For any $f \in C([a, b])$, let $I_h f \in C([a, b])$ be such that $I_h f \in \mathcal{P}_1$ on each $[x_{j-1}, x_j]$ ($j = 1, \dots, N$) and $I_h f(x_j) = f(x_j)$ ($j = 0, \dots, N$). Prove the following:

- (1) If $f \in C^2([a, b])$ and $M_2 = \max_{a \leq x \leq b} |f''(x)|$, then

$$\max_{a \leq x \leq b} |f(x) - (I_h f)(x)| \leq \frac{M_2}{8} h^2;$$

- (2) If $f \in C^3([a, b])$, $M_3 = \max_{a \leq x \leq b} |f'''(x)|$, and $m_j = (x_{j-1} + x_j)/2$ ($j = 1, \dots, N$), then

$$\max_{1 \leq j \leq N} |f'(m_j) - (I_h f)'(m_j)| \leq \frac{M_3}{24} h^2.$$

6. Let $n = 1$ and $a = x_0 < x_1 = b$. Show that the Peano kernel $K_1(x, t)$ is given by

$$K_1(x, t) = \begin{cases} (t - a)(x - b)/(b - a) & \text{if } a \leq t \leq x \leq b, \\ (x - a)(t - b)/(b - a) & \text{if } a \leq x \leq t \leq b. \end{cases}$$

Let $h \in C([a, b])$ and define

$$u(x) = \int_a^b K_1(x, t) h(t) dt \quad \forall x \in [a, b].$$

Show that $u''(x) = h(x)$ if $a < x < b$ and $u(a) = u(b) = 0$.