## Math 270B: Numerical Analysis (Part B) <br> Winter quarter 2023

## Homework Assignment 8

Due: 10:00 am, Friday, March 10, 2023.

1. Given a numerical integration formula on $[-1,1]$

$$
\begin{equation*}
\int_{-1}^{1} g(t) d t \approx \sum_{j=1}^{n} a_{j} g\left(t_{j}\right) \tag{1}
\end{equation*}
$$

Define for an interval $[a, b]$ that $A_{j}=(b-a) a_{j} / 2$ and $x_{j}=\left[(b-a) t_{j}+a+b\right] / 2, j=$ $1, \ldots, n$. Show that the numerical integration formula on $[a, b]$

$$
\int_{a}^{b} f(x) d x \approx \sum_{j=1}^{n} A_{j} f\left(x_{j}\right)
$$

has the same degree of precision as that of the formula (1).
2. Find $A, B, C$ such that the weighted numerical quadrature

$$
\int_{-2}^{2}|x| f(x) d x \approx A f(-1)+B f(0)+C f(1)
$$

is exact for polynomials of degree as high as possible. Find the degree of precision of the quadrature.
3. Consider an interpolatory quadrature

$$
\int_{a}^{b} f(x) d x \approx \sum_{k=0}^{n} A_{k} f\left(x_{k}\right)
$$

Define for each integer $j \geq 0$

$$
F_{j}(t)=\int_{a}^{b}(x-t)_{+}^{j} d x-\sum_{k=0}^{n} A_{k}\left(x_{k}-t\right)_{+}^{j}
$$

Show that

$$
\int_{a}^{b} F_{j}(t) d t=0, \quad j=0, \ldots, n-1
$$

4. Consider the trapezoidal formula

$$
\int_{a}^{b} f(x) d x \approx \frac{1}{2}(b-a)[f(a)+f(b)] .
$$

(1) Show that the degree of precision of the formula is $m=1$.
(2) Calculate the Peano kernel $K_{1}$ of the formula and show that it does not change sign in $[a, b]$.
(3) Let $f \in C^{2}([a, b])$. Show that there exists $\xi \in(a, b)$ such that

$$
\int_{a}^{b} f(x) d x-\frac{1}{2}(b-a)[f(a)+f(b)]=-\frac{1}{12}(b-a)^{3} f^{\prime \prime}(\xi)
$$

(4) Let $N \geq 1$ be an integer, $h=(b-a) / N$, and $x_{j}=a+j h, j=0, \ldots, N$. Let $f \in C^{2}([a, b])$. Prove that there exists $\eta \in(a, b)$ such that

$$
\int_{a}^{b} f(x) d x-\left\{\frac{h}{2}[f(a)+f(b)]+h \sum_{j=1}^{N-1} f\left(x_{j}\right)\right\}=-\frac{(b-a) f^{\prime \prime}(\eta)}{12} h^{2}
$$

5. Consider the Newton-Cotes formula

$$
\int_{a}^{b} f(x) d x \approx \sum_{j=0}^{n} A_{j} f\left(x_{j}\right)
$$

with $n+1$ points $x_{j}=a+j(b-a) / n, j=0, \ldots, n$.
(1) Show that $A_{j}=A_{n-j}$ for $j=0, \ldots,[n / 2]$.
(2) Show that the degree of precision of the formula is $n$ if $n$ is odd and is $n+1$ if $n$ is even.

