Math 270B: Numerical Analysis (Part B) Winter quarter 2023 Homework Assignment 8 Due: 10:00 am, Friday, March 10, 2023.

1. Given a numerical integration formula on [-1, 1]

$$\int_{-1}^{1} g(t) dt \approx \sum_{j=1}^{n} a_j g(t_j).$$
 (1)

Define for an interval [a, b] that $A_j = (b - a)a_j/2$ and $x_j = [(b - a)t_j + a + b]/2$, $j = 1, \ldots, n$. Show that the numerical integration formula on [a, b]

$$\int_{a}^{b} f(x) \, dx \approx \sum_{j=1}^{n} A_j f(x_j)$$

has the same degree of precision as that of the formula (1).

2. Find A, B, C such that the weighted numerical quadrature

$$\int_{-2}^{2} |x| f(x) \, dx \approx Af(-1) + Bf(0) + Cf(1)$$

is exact for polynomials of degree as high as possible. Find the degree of precision of the quadrature.

3. Consider an interpolatory quadrature

$$\int_{a}^{b} f(x) \, dx \approx \sum_{k=0}^{n} A_k f(x_k).$$

Define for each integer $j \ge 0$

$$F_j(t) = \int_a^b (x-t)_+^j dx - \sum_{k=0}^n A_k (x_k - t)_+^j.$$

Show that

$$\int_{a}^{b} F_{j}(t)dt = 0, \qquad j = 0, \dots, n - 1.$$

4. Consider the trapezoidal formula

$$\int_{a}^{b} f(x) \, dx \approx \frac{1}{2} (b-a) \left[f(a) + f(b) \right].$$

- (1) Show that the degree of precision of the formula is m = 1.
- (2) Calculate the Peano kernel K_1 of the formula and show that it does not change sign in [a, b].
- (3) Let $f \in C^2([a, b])$. Show that there exists $\xi \in (a, b)$ such that

$$\int_{a}^{b} f(x) \, dx - \frac{1}{2}(b-a) \left[f(a) + f(b) \right] = -\frac{1}{12}(b-a)^{3} f''(\xi).$$

(4) Let $N \ge 1$ be an integer, h = (b-a)/N, and $x_j = a + jh$, j = 0, ..., N. Let $f \in C^2([a,b])$. Prove that there exists $\eta \in (a,b)$ such that

$$\int_{a}^{b} f(x) \, dx - \left\{ \frac{h}{2} \left[f(a) + f(b) \right] + h \sum_{j=1}^{N-1} f(x_j) \right\} = -\frac{(b-a)f''(\eta)}{12} h^2,$$

5. Consider the Newton–Cotes formula

$$\int_{a}^{b} f(x) \, dx \approx \sum_{j=0}^{n} A_{j} f(x_{j})$$

with n + 1 points $x_j = a + j(b - a)/n, j = 0, ..., n$.

- (1) Show that $A_j = A_{n-j}$ for j = 0, ..., [n/2].
- (2) Show that the degree of precision of the formula is n if n is odd and is n + 1 if n is even.