Math 270B: Numerical Analysis (Part B)
Winter quarter 2023

Homework Assignment 9

Due: 5:00 pm, Friday, March 17, 2023.

1. (1) Show for any $f \in C([0,1])$ and any integer $n \geq 1$ that

$$\int_0^1 B_n f(x)\, dx = \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right),$$

where $B_n f$ is the nth Bernstein polynomial of f.

(2) Show that the degree of precision of the following numerical quadrature is $m = 1$:

$$\int_0^1 f(x)\, dx \approx \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right)$$

(3) Show that

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) = \int_0^1 f(x)\, dx \qquad \forall f \in C([0,1]).$$

2. Let $p_3 \in \mathcal{P}_3$ be the Hermite interpolation polynomial of $f \in C^1([a,b])$ determined by

$$p_3(a) = f(a), \quad p_3'(a) = f'(a), \quad p_3(b) = f(b), \quad p_3'(b) = f'(b).$$

(1) Show that

$$\int_a^b p_3(x)\, dx = \frac{1}{2}(b-a)[f(a)+f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)].$$

(2) Determine the degree of precision of the numerical quadrature

$$\int_a^b f(x)\, dx \approx \frac{1}{2}(b-a)[f(a)+f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)].$$

3. Let $f : [a,b] \to \mathbb{R}$ be integrable over $[a,b]$. Let $n \geq 1$ be an integer, $h = (b-a)/(2n)$, and $x_i = a + ih$ $(i = 0, \dots, 2n)$.

(1) Derive the composite Simpson's formula for the integration of a function of f over $[a,b]$ by applying the basic Simpson's formula to each of the subinterval $[x_{2i-2}, x_{2i}]$ $(i = 0, \dots, n)$.

(2) Assume $f \in C^4([a,b])$. Derive an error formula for the composite Simpson's formula that is derived in Part (1).

4. Let $\{Q_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials on $[a, b]$. Fix $n \geq 1$. Let $x_1, \ldots, x_n$ be the n distinct roots of Q_n in (a, b). Let

$$\int_a^b f(x)\, dx \approx \sum_{j=1}^{n} A_j f(x_j)$$

be the corresponding Gaussian quadrature. Show that

$$\sum_{j=1}^{n} A_j Q_k(x_j) = 0, \qquad k = 1, \ldots, 2n - 1.$$

5. Consider a Gaussian formula

$$\int_a^b f(x)\, dx \approx \sum_{j=1}^{n} A_j f(x_j).$$

Show that for any $f \in C([a, b])$ the error

$$e_n(f) = \int_a^b f(x)\, dx - \sum_{j=1}^{n} A_j f(x_j)$$

satisfies

$$|e_n(f)| \leq 2(b - a) \min_{q \in \mathcal{P}_{2n-1}} \|f - q\|_{C([a,b])}.$$

6. Let $n \geq 1$ be an integer. The Gauss–Chebyshev quadrature is the weighted Gaussian quadrature on $[-1, 1]$ with the weight $1/\sqrt{1 - x^2}$ using

$$x_j = \cos(2j - 1)\pi/2n \qquad (j = 1, \ldots, n),$$

the n roots of the nth Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. Show that the Gauss–Chebyshev formula is given by

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} dx \approx \frac{\pi}{n} \sum_{j=1}^{n} f(x_j).$$

7. Let $f \in C([a, b])$ and denote by $I(f)$ the integral of f over $[a, b]$. Let $N \geq 1$ be an integer, $h = (b - a)/2N$, and $x_j = a + jh$, $j = 0, \ldots, 2N$. Let T_N, T_{2N}, and S_N denote, respectively, the approximate value of $I(f)$ by the composite trapezoidal rule with N subintervals $[x_{2j-1}, x_{2j}]$, $j = 1, \ldots, N$, by the composite trapezoidal rule with $2N$ subintervals $[x_{j-1}, x_j]$, $j = 1, \ldots, 2N$, and by the composite Simpson rule with N subintervals $[x_{2j-1}, x_{2j}]$, $j = 1, \ldots, N$. Prove that the Richardson extrapolation using T_N and T_{2N} leads to exactly S_N, i.e.,

$$S_N = \frac{4T_{2N} - T_N}{3}.$$