

Friday. 1/13/2023

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① HW1 due Wed. 1/18 by 10:00am
upload HW to grade scope.

② Today: ③ Newton's method for systems of equations

④ Contracting mappings and
Banach's fixed-point theorem

Newton's iteration

$$f(x) = 0, f(x) = \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix}, x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \in E, E \subseteq \mathbb{R}^n: \text{open}.$$

$$f: E \rightarrow \mathbb{R}^n. \text{ Assume: } x^* \in E, f(x^*) = 0.$$

The Jacobian matrix of f (if all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist) is $Df(x) = \left[\frac{\partial f_i}{\partial x_j} \right]$

$$y = f(x) = f(x^{(k)}) + Df(x^{(k)})(x - x^{(k)}) + \dots$$

$$y=0 \Rightarrow x=x^{(k+1)}. 0 = f(x^{(k)}) + Df(x^{(k)})(x^{(k+1)} - x^{(k)})$$

Newton's iteration:

$$\boxed{x^{(k+1)} = x^{(k)} - [Df(x^{(k)})]^{-1} f(x^{(k)})}, k=0, 1, \dots$$

We shall denote by $\|\cdot\|$ a norm on \mathbb{R}^n and also the induced matrix norm on $\mathbb{R}^{n \times n}$, the space of all $n \times n$ real matrices. Recall $\overline{B(x^*, \delta)} = \{x \in \mathbb{R}^n : \|x - x^*\| \leq \delta\}$.

Theorem Let $E \neq \emptyset$ be an open subset of \mathbb{R}^n . Assume $f \in C^1(E, \mathbb{R}^n)$ and f has a unique root $x^* \in E$: $f(x^*) = 0$. Assume there exist $\delta > 0$, $\rho > 0$, and $L > 0$ satisfying:

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- (1) The closed ball $\overline{B(x^*, \delta)} \subseteq E$ and $\delta \leq \frac{\rho}{L}$, [2]
- (2) $Df(x)$ is nonsingular and $\|[Df(x)]^{-1}\| \leq \frac{1}{\rho}$
for any $x \in \overline{B(x^*, \delta)}$; and
- (3) $\|Df(x) - Df(y)\| \leq L \|x - y\| \quad \forall x, y \in \overline{B(x^*, \delta)}$. Then,
for any $x^{(0)} \in B(x^*, \delta)$, Newton's iteration
- $$x^{(k+1)} = x^{(k)} - [Df(x_k)]^{-1} f(x_k) \quad (k=0, 1, \dots)$$
- Satisfies $\|x^{(k+1)} - x^{(0)}\| \leq \frac{L}{2\rho} \|x^{(k)} - x^{(0)}\|^2 \quad (k=0, 1, \dots)$.

The proof of theorem is similar to that for single equations. But, here are a few points that are different.

- ① $f(x) - f(y) = \int_0^1 g'(t) dt = \int_0^1 Df(tx + (1-t)y)(x-y) dt$
where $g(t) = f(tx + (1-t)y)$.
- ② If A_1, A_2 are $n \times n$ matrices and $x \in \mathbb{R}^n$ then
 $\|A_1 A_2\| \leq \|A_1\| \|A_2\|$ and $\|A_1 x\| \leq \|A_1\| \|x\|$ since
the matrix norm is induced from the vector norm.

Contracting Mappings

Definition A mapping $\Phi: E(\subseteq \mathbb{R}^n) \rightarrow \mathbb{R}^n$ is a contracting mapping with respect to a norm $\|\cdot\|$ on \mathbb{R}^n , if there exists $L \in (0, 1)$ such that

$$\|\Phi(x) - \Phi(y)\| \leq L \|x - y\| \quad \forall x, y \in E. \quad (*)$$

Example If $g: \mathbb{R} \rightarrow \mathbb{R}$ satisfies $g'(x)$ exists and

$|g'(x_1)| \leq L$ for some $L \in (0, 1)$ then g is contracting. [3]

$$|g(x_1) - g(y_1)| = |g'(s)(x_1 - y_1)| = |g'(s)| |x_1 - y_1| \leq L |x_1 - y_1|.$$

Banach's Fixed Point Theorem Let $E \subseteq \mathbb{R}^n$ be a closed subset and $\Phi: E \rightarrow E$ a contracting mapping as in (*). Then there exists a unique $x^* \in E$ such that $\Phi(x^*) = x^*$. Moreover, for any $x_0 \in E$, define $x_{k+1} = \Phi(x_k)$ ($k = 0, 1, \dots$). Then $x_k \rightarrow x^*$ and

$$\|x_k - x^*\| \leq \frac{L}{1-L} \|x_k - x_{k-1}\| \leq \frac{L^k}{1-L} \|x_1 - x_0\|, \quad k=1, 2, \dots$$

Proof Uniqueness of x^* : If $\Phi(x^*) = x^*$, $\Phi(y^*) = y^*$ ($x^*, y^* \in E$). Then

$$\|x^* - y^*\| = \|\Phi(x^*) - \Phi(y^*)\| \leq L \|x^* - y^*\|$$

$$0 \leq (1-L) \|x^* - y^*\| \leq 0 \implies \|x^* - y^*\| = 0 \implies x^* = y^*$$

Existence and convergence. Let $x_0 \in E$ and define $x_{k+1} = \Phi(x_k)$ ($k = 0, 1, \dots$). Observe:

$$j \geq 1: \quad \|x_{j+1} - x_j\| = \|\Phi(x_j) - \Phi(x_{j-1})\| \leq L \|x_j - x_{j-1}\|.$$

Now for any $k \geq 1, m \geq 1$.

$$\|x_{k+m} - x_k\| = \|(x_{k+m} - x_{k+m-1}) + (x_{k+m-1} - x_{k+m-2}) + \dots + (x_{k+1} - x_k)\|$$

$$\leq \|x_{k+m} - x_{k+m-1}\| + \|x_{k+m-1} - x_{k+m-2}\| + \dots + \|x_{k+1} - x_k\|$$

$$\|x_{k+1} - x_k\| = \|x_{k+1} - x_k\|$$

$$\|x_{k+2} - x_{k+1}\| \leq L \|x_{k+1} - x_k\|$$

$$\|x_{k+3} - x_{k+2}\| \leq L \|x_{k+2} - x_{k+1}\| \leq L^2 \|x_{k+1} - x_k\|$$

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$$\|x_{k+m} - x_{k+m-1}\| \leq L^{m-1} \|x_{k+1} - x_k\|$$

$$\|x_{k+m} - x_k\| \leq (1 + L + \dots + L^{m-1}) \|x_{k+1} - x_k\|$$

$$= \frac{1-L^m}{1-L} \|x_{k+1} - x_k\| \leq \frac{1}{1-L} \|x_{k+1} - x_k\|$$

$$\leq \frac{L}{1-L} \|x_k - x_{k-1}\| \leq \frac{L^k}{1-L} \|x_1 - x_0\| \rightarrow 0 \text{ as } k \rightarrow \infty.$$

So $\{x_k\}$ is a Cauchy sequence. $x_k \rightarrow x^* \in \mathbb{R}^n$

But E is closed so $x^* \in E$. Since $\Phi(x_k) = \underline{\Phi}(x_k)$ and $\underline{\Phi}$ is continuous, we have $x^* = \underline{\Phi}(x^*)$. QED

Discussions

① Given $\varepsilon > 0$, determine k such that $\|x_k - x^*\| < \varepsilon$.

$$\|x_k - x^*\| \leq \frac{L^k}{1-L} \|x_1 - x_0\| < \varepsilon \Rightarrow L^k < (1-L) \varepsilon / \|x_1 - x_0\|$$

$$k > \frac{\ln((1-L)\varepsilon / \|x_1 - x_0\|)}{\ln L}. \quad (\text{if } x_1 \neq x_0).$$

② When $\|\underline{\Phi}(x) - \underline{\Phi}(y)\| \leq L \|x - y\|$?

If $\underline{\Phi}$ is a C^1 -mapping: $\underline{\Phi}(x) = \begin{bmatrix} \underline{\Phi}_1(x) \\ \vdots \\ \underline{\Phi}_n(x) \end{bmatrix}$ all $\underline{\Phi}_i$ are C^1 .

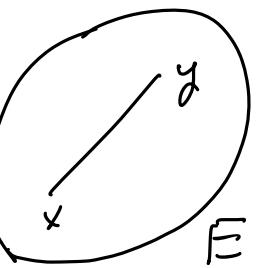
and E is convex then

$$\|D\underline{\Phi}(x)\| \leq L, \forall x \in E \Rightarrow \|\underline{\Phi}(x) - \underline{\Phi}(y)\| \leq L \|x - y\|. \quad \forall x, y \in E$$

Proof Fix $x \neq y$. Define $g(t) = \underline{\Phi}(tx + (1-t)y)$ ($0 \leq t \leq 1$)

$$g: [0, 1] \rightarrow \mathbb{R}^n$$

$$\begin{aligned} \underline{\Phi}(x) - \underline{\Phi}(y) &= g(1) - g(0) = \int_0^1 \frac{d}{dt} g(t) dt \\ &= \int_0^1 D\underline{\Phi}(tx + (1-t)y)(x-y) dt \end{aligned}$$



$$\|\underline{\Phi}(x) - \underline{\Phi}(y)\| \leq \left\| \int_0^1 D\underline{\Phi}(tx + (1-t)y)(x-y) dt \right\|$$

$$\leq \int_0^1 \|D\underline{\Phi}(\dots)(x-y)\| dt \leq \int_0^1 \|D\underline{\Phi}(\dots)\| \|x-y\| dt \leq L \|x-y\|$$

QED