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Weierstrass Theorem $\forall f \in C([a,b]) \quad \forall \varepsilon > 0 \quad \exists p \in \mathcal{P} \text{ s.t.}$
 $\|f - p\| := \max_{a \leq x \leq b} |f(x) - p(x)| < \varepsilon.$

Equivalently, $\forall f \in C([a,b])$ there exist $p_k \in \mathcal{P}$ ($k=1, 2, \dots$) s.t.

$$\lim_{k \rightarrow \infty} \|f - p_k\| = 0.$$

($\deg p_k$ may not be $\leq k$)

Fix $n \geq 0$: integer. $\forall f \in C([a,b])$ define $E_n(f) = \inf_{p \in \mathcal{P}_n} \|f - p\|$.

① $\|f\| \geq E_0(f) \geq E_1(f) \geq \dots \geq E_n(f) \geq \dots \geq 0$

② $\lim_{n \rightarrow \infty} E_n(f) = 0 \quad (\iff \text{W-Thm})$

Best unif. approx. of $f \in C([a,b])$ in \mathcal{P}_n : $\exists ! p_n \in \mathcal{P}_n$ s.t.

$$\|f - p_n\| = E_n(f) \leq \|f - q_n\| \quad \forall q_n \in \mathcal{P}_n.$$

Chebyshev Alternation Theorem Let $f \in C([a,b]) \setminus \mathcal{P}_n$. $p \in \mathcal{P}_n$ is a best unif. approx of f in $\mathcal{P}_n \iff \exists a \leq x_1 < \dots < x_{n+1} \leq b$ s.t. for $e = f - p$: $|e(x_k)| = \|e\| \quad (1 \leq k \leq n+1)$; $e(x_k) e(x_{k+1}) < 0 \quad (1 \leq k \leq n+1)$.

How fast does $E_n(f) \rightarrow 0$?

Bernstein polynomials: $(B_n f)(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} \quad x \in [0,1]$

[For $0 < x < 1$, $(B_n f)(x) := \mathbb{E} f\left(\frac{S_n}{n}, x\right)$, $S_{n,x} \in \{0, 1, \dots, n\}$ is a random variable with binomial distribution $B(n, x)$. $P(S_{n,x}=k) = \binom{n}{k} x^k (1-x)^{n-k}$]
- $\mathbb{E} S_{n,x} = nx$, $\text{Var}(S_{n,x}) = nx(1-x)$.

Theorem If $f \in \text{Lip}[0,1]$, then $\|B_n f - f\| \leq O\left(\frac{1}{\sqrt{n}}\right)$ as $n \rightarrow \infty$.

Note: $f \in \text{Lip}[0,1]$: If $\exists L > 0$ s.t. $|f(x) - f(y)| \leq L|x-y|$.

This result is optimal due to the following example:

Example If $f(x) = |x - \frac{1}{2}|$, then $f \in \text{Lip}[0,1]$. It's proved that

$$\frac{1}{2\sqrt{n}} \leq \|B_n f - f\|_{C([0,1])} \leq \frac{2}{3\sqrt{n}}, \quad n=1, 2, \dots$$

Theorem (Jackson) If $p \in \mathcal{N}$ and $f \in C^p([a,b])$ then

$$E_n(f) \leq O\left(\frac{1}{n^p}\right) \text{ as } n \rightarrow \infty.$$