Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 1

Due: 1:00 pm, Wednesday, January 17, 2024.

1. Let $a, b \in \mathbb{R}$ with $a < b$. Let $f \in C^2([a,b])$ be such that $f(a) < 0$ and $f(b) > 0$, $\min_{x\in[a,b]} f'(x) > 0$, and $\min_{x\in[a,b]} f''(x) > 0$. Denote by $x^* \in (a,b)$ the unique root of f on $[a,b]$. Suppose $x_0 \in (x^*, b)$. Let x_n $(n = 1, 2, \dots)$ be the sequence of Newton iterates for solving $f(x) = 0$. Prove that $x^* < x_{n+1} < x_n$ for all $n = 0, 1, \dots$ and that $x_n \to x^*$.

2. The function $f(x) = x/\sqrt{1+x^2}$ has the unique zero $x^* = 0$ in $\mathbb{R}$. Suppose we solve the equation $f(x) = 0$ using Newton's method: Set $x_0 \in \mathbb{R}$ and define $x_{n+1} = x_n - f(x_n)/f'(x_n)$ $(n = 0, 1, \dots)$. Prove that $x_n \to x^* = 0$ if $|x_0| < 1$ and $|x_n| \to \infty$ if $|x_0| > 1$.

3. Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by
$$F(x,y) = \begin{bmatrix} e^{x^2+y^2} - 1 \\ x + y + 2\cos(x+y) \end{bmatrix}.$$
Calculate the Jacobian matrix $\nabla F(x,y)$ and find all (x,y) such that $\nabla F(x,y)$ are singular.

4. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be continuously differentiable. Consider a vector norm $\|\cdot\|$ on $\mathbb{R}^n$ and its induced matrix norm, also denoted by $\|\cdot\|$. Prove the following:
   (1) If $x, y \in \mathbb{R}^n$ and $K := \sup_{0\le t\le 1} \|DF(tx + (1-t)y)\|$, then $\|F(x) - F(y)\| \le K\|x - y\|$.
   (2) If there exists $L > 0$ such that $\|DF(x) - DF(y)\| \le L\|x - y\|$ for all $x, y \in \mathbb{R}^n$, then
   $$\|F(x) - F(y) - DF(y)(x-y)\| \le \frac{L}{2}\|x - y\|^2 \qquad \forall x, y \in \mathbb{R}^n.$$

5. Show that the the function $f(x) = \cos x$ has a unique fixed point x^* and that the fixed-point iteration $x_{n+1} = \cos x_n$ $(n = 0, 1, \dots)$ converges (i.e., $\lim_{n\to\infty} x_n = x^*$) for any $x_0 \in \mathbb{R}$.

6. Let $f \in C^2(\mathbb{R})$. Assume f has a unique zero x^* such that $0 < f'(x^*) < 2$. Define $\phi(x) = x - f(x)$ for all $x \in \mathbb{R}$ and
$$\psi(x) = \frac{x\phi(\phi(x)) - (\phi(x))^2}{\phi(\phi(x)) - 2\phi(x) + x} \qquad \text{if } x \ne x^* \qquad \text{and} \qquad \psi(x^*) = \lim_{x\to x^*} \psi(x).$$
Prove the following:
   (1) The zero x^* of f is the unique fixed point for both ϕ and ψ.
   (2) If $x_0 \in \mathbb{R}$ is sufficiently close to x^* and $x_{n+1} = \phi(x_n)$ $(n = 0, 1, \dots)$, then $x_n \to x^*$ linearly, i.e., there exists $C_1 \in (0,1)$ such that $|x_{n+1} - x^*| \le C_1|x_n - x^*|$ for all n.
   (3) If $x_0 \in \mathbb{R}$ is sufficiently close to x^* and $x_{n+1} = \psi(x_n)$ $(n = 0, 1, \dots)$, then $x_n \to x^*$ quadratically, i.e., there exists $C_2 > 0$ such that $|x_{n+1} - x^*| \le C_2|x_n - x^*|^2$ for all n.
   (4) The iteration $x_{n+1} = \psi(x_n)$ $(n = 0, 1, \dots)$ is exactly the quasi-Newton iteration
   $$x_{n+1} = x_n - \frac{[f(x_n)]^2}{f(x_n) - f(x_n - f(x_n))}, \qquad n = 0, 1, \dots.$$