Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 2

Due: 1:00 pm, Wednesday, January 24, 2024.

1. Let $f \in C^2(\mathbb{R}^n)$ and assume it has a unique critical point.
   (1) Write down Newton's iteration scheme for finding the critical point of f.
   (2) State pricisely a local convergence thereom for this scheme. You may add some more assumptions on f.

2. Let $f \in C^1(\mathbb{R}^n)$, $x^{(0)} \in \mathbb{R}^n$, and $x^{(k+1)} = x^{(k)} - \alpha_k \nabla f(x^{(k)})$, where

$$\alpha_k = \arg\min\{f(x^{(k)} - \alpha \nabla f(x^{(k)})) : \alpha \geq 0\} > 0 \qquad (k = 0, 1, \dots).$$

   Show for each k that $\nabla f(x^{(k)}) \cdot \nabla f(x^{(k+1)}) = 0$.

3. Let A be an $n \times n$ symmetric positive definite matrix, $b \in \mathbb{R}^n$, and $f(x) = (1/2)x^T Ax - b^T x$ for any $x \in \mathbb{R}^n$. Let x^* be the unique minimum point of the convex quadratic function f. The gradient descent method applied to minimize $f(x)$ produces $x^{(k+1)} = x^{(k)} - \alpha_k \nabla f(x^{(k)})$ $(k = 0, 1, \dots)$, where $x^{(0)} \in \mathbb{R}^n$ and $\alpha_k = \arg\min\{f(x^{(k)} - \alpha \nabla f(x^{(k)})) : \alpha \geq 0\}$.
   (1) Prove that $\alpha_k = \nabla f(x^{(k)})^T \nabla f(x^{(k)}) / (\nabla f(x^{(k)})^T A \nabla f(x^{(k)})$ for any k with $x^{(k)} \neq x^*$.
   (2) Denote by λ_{min} and λ_{max} the smallest and largest eigenvalues of A, respectively, and denote $\beta = (\lambda_{max} - \lambda_{min})/(\lambda_{max} + \lambda_{min}) \in [0, 1)$. Use the Kantorovich inequality

$$\frac{(y^T Ay)(y^T A^{-1} y)}{(y^T y)^2} \leq \frac{(\lambda_{min} + \lambda_{max})^2}{4\lambda_{min}\lambda_{max}} \qquad \forall y \in \mathbb{R}^n \text{ with } y \neq 0$$

   to prove $f(x^{(k+1)}) - f(x^*) \leq \beta^2 (f(x^{(k)}) - f(x^*))$ $(k = 0, 1, \dots)$.
   (3) Prove that $x_k \to x^*$ (at least) linearly.

4. Let $\alpha_i > 0$, $c_i \in \mathbb{R}$, $a_i \in \mathbb{R}$ $(i = 1, \dots, n)$, and $b \in \mathbb{R}$. Assume at least one of $a_1, \dots, a_n$ is nonzero. Minimize $Q(x_1, \dots, x_n) = \sum_{i=1}^n (x_i - c_i)^2/\alpha_i$ subject to $\sum_{i=1}^n a_i x_i = b$.

5. (1) Maximize $H(p_1, \dots, p_n) = -\sum_{k=1}^n p_k \ln p_k$ for all $p_k > 0$ subject to $\sum_{k=1}^n p_k = 1$.
   (2) Let $\mu > 0$. Maximize $H(p_1, p_2, \dots) = -\sum_{k=1}^\infty p_k \ln p_k$ for $p_k > 0$ $(k = 1, 2, \dots)$ subject to $\sum_{k=1}^\infty p_k = 1$ and $\sum_{k=1}^\infty k p_k = \mu$.

6. Let $X_1, \dots, X_n \in \mathbb{R}$ be independent random variables with the expection $E(X_i) = \mu$ and variance $\mathrm{Var}(X_i) = \sigma^2$ $(i = 1, \dots, n)$ for some $\mu \in \mathbb{R}$ and $\sigma > 0$. Find the coefficients $a_1, \dots, a_n$ that minimize $\mathrm{Var}(\sum_{i=1}^n a_i X_i)$ subject to $E(\sum_{i=1}^n a_i X_i) = \mu$.

7. Let $E \subset \mathbb{R}^n$ be open and bounded, $f \in C(\overline{E})$, and $h \in C(\overline{E}, \mathbb{R}^m)$ for some positive integers n and m. (An overline denotes the closure.) Assume $\{x \in \overline{E} : h(x) = 0\} \neq \emptyset$. Let M_k be an increasing sequence of positive numbers such that $M_k \to \infty$. Assume for each $k \geq 1$ that $x_k \in \overline{E}$ is a global minimizer of the penalty function $f(x) + (M_k/2)\|h(x)\|_2^2$ over $\overline{E}$. Prove that any cluster point x^* of $\{x^{(k)}\}$ is a global minimizer of f on $\overline{E}$ with the constraint $h(x) = 0$.