Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 3

Due: 1:00 pm, Wednesday, January 31, 2024.

1. (1) Let $f(x) = 1/x$ $(0 < x < 1)$. Prove that there exists no polynomial p such that

$$\sup_{0<x<1} |f(x) - p(x)| < 1.$$

   (2) Let $f(x) = \sin(1/x)$ $(0 < x < 1)$. Prove that there exists no $g \in C([0,1])$ such that

$$\sup_{0<x<1} |f(x) - g(x)| < 1.$$

2. Let $B_n f \in \mathcal{P}_n$ $(n = 0, 1, \dots)$ be the Bernstein polynomials of $f \in C([0,1])$.
   (1) Let $p_2(x) = x^2$ and $n \geq 2$. Show that $B_n p_2(x) = ((n-1)/n)x^2 + (1/n)x$ for all $x \in [0,1]$.
   (2) In general, is $B_n f \in \mathcal{P}_n$ the best uniform approximation of $f \in C([0,1])$ in $\mathcal{P}_n$ on $[0,1]$?
   (3) If $f \in C^1([0,1])$, then $\|(B_n f)' - f'\|_{C([a,b])} \to 0$ as $n \to \infty$.

3. Let $k \geq 1$ be an integer, $f \in C^k([a,b])$, and $\epsilon > 0$. Show that there exists $p \in \mathcal{P}$ such that

$$\|f - p\|_{C([a,b])} < \epsilon, \qquad \|f' - p'\|_{C([a,b])} < \epsilon, \qquad \dots, \qquad \|f^{(k)} - p^{(k)}\|_{C([a,b])} < \epsilon.$$

4. Let $f \in C([a,b])$ but $f \notin \mathcal{P}$. Show that there exits no polynomial $p \in \mathcal{P}$ such that

$$\|f - p\|_{C([a,b])} \leq \|f - q\|_{C([a,b])} \qquad \forall q \in \mathcal{P}.$$

5. Let $f \in C([a,b])$ and $q_n \in \mathcal{P}_n$ for some $n \geq 0$. Let $p_n \in \mathcal{P}_n$ be the best uniform approximation of f in $\mathcal{P}_n$. Prove that $p_n + q_n$ is the best uniform approximation of $f + q_n$ in $\mathcal{P}_n$.

6. Let $c > 0$. Let $f \in C([-c,c])$ be an even (odd) function. Show that the best uniform approximation of f in $\mathcal{P}_n$ for an integer $n \geq 0$ is also an even (odd) function.

7. Show that $p_1(x) = x - 1/8$ is the best uniform approximation of $f(x) = x^2$ in $\mathcal{P}_1$ on $[0,1]$.

8. Let $f(x) = x^4$ $(0 \leq x \leq 1)$. Find the best uniform approximation of f in $\mathcal{P}_1$ on $[0,1]$.