Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 4

Due: 10:00 pm, Thursday 8, 2024.

1. Let $f \in C([a, b])$ and define
$$\mu_n(f) = \int_a^b x^n f(x)\, dx, \qquad n = 0, 1, \ldots .$$
Show that $f(x) = 0$ for all $x \in [a, b]$ if and only if $\mu_n(f) = 0$ for all $n = 0, 1, \ldots .$

2. Let $n \geq 0$ be an integer and T_n the n-th Chebyshev polynomial. Show that
$$\int_{-1}^{1} [T_n(x)]^2 = 1 - \frac{1}{4n^2 - 1}.$$

3. Let $n \geq 0$ be an integer and T_n the nth Chebyshev polynomial of first kind. Let $P \in \mathcal{P}_n$ satisfy that $|P(x)| \leq 1$ for all $x \in [-1, 1]$. Show that
$$|P(y)| \leq |T_n(y)| \qquad \forall y \notin [-1, 1].$$

4. Let $f \in C_{2\pi}$ and $n \geq 0$ be an integer. Prove that there exists $T_n \in \mathcal{T}_n$ such that
$$\|f - T_n\|_{C_{2\pi}} \leq \|f - S_n\|_{C_{2\pi}} \qquad \forall S_n \in \mathcal{T}_n.$$

5. Given any function g on $[a, b]$, define
$$g^*(\theta) = g\left(\frac{(b-a)cos\theta + (a+b)}{2}\right) \qquad \forall \theta \in (-\infty, \infty).$$
Let $f \in C([a, b])$ and $n \geq 0$ be an integer. Let $p \in \mathcal{P}_n$ and $T \in \mathcal{T}_n$ satisfy
$$\|f - p\|_{C([a,b])} = E_n(f) := \min_{q \in \mathcal{P}_n} \|f - q\|_{C([a,b])},$$
$$\|f^* - T\|_{C_{2\pi}} = E_n^*(f^*) := \min_{S \in \mathcal{T}_n} \|f^* - S\|_{C_{2\pi}}.$$
Show that $E_n(f) = E_n^*(f^*)$ and that $T = p^*$.

6. Define $\chi(x) = -1$ if $-1 \leq x < 0$ and $\chi(x) = 1$ if $0 \leq x \leq 1$.
   1. Show that $\inf_{f \in C([-1,1])} \sup_{-1 \leq x \leq 1} |f(x) - \chi(x)| = 1$, and that there exist infinitely many $f \in C([-1, 1])$ such that $\sup_{-1 \leq x \leq 1} |f(x) - \chi(x)| = 1$.
   2. Show that
   $$\inf_{f \in C([-1,1])} \int_{-1}^{1} |f(x) - \chi(x)|^2 dx = 0,$$
   and that there exists no $f \in C([-1, 1])$ such that
   $$\int_{-1}^{1} |f(x) - \chi(x)|^2 dx = 0.$$

7. Let $a, b \in \mathbb{R}$ with $a < b$, $f \in C([a, b])$, and $\varepsilon > 0$. Show that there exists a polynomial p such that $\|f - p\|_{L^2(a,b)} < \varepsilon$.

8. Find the least-squares approximation of $f(x) = x^4$ in $\mathcal{P}_1$ over $[0, 1]$.