Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 5

Due: 1:00 pm, Friday, February 16, 2024.

1. Let $p(x) = \sum_{k=0}^{n} a_k x^k \in \mathcal{P}_n$ be the least-squares approximation of a given $f \in C([0,1])$ over $[0,1]$. Find the coefficient matrix of the linear system that determines $a_0, \dots, a_n$.

2. Given $q_1, \dots, q_m \in \mathcal{P}$. Prove that they are linearly independent in $\mathcal{P}$ if and only if the Gram matrix $G(q_1, \dots, q_m) = [G_{ij}]_{i,j=1}^{m}$ is symmetric positive definite, where
$$G_{ij} = \int_a^b p_i(x) p_j(x)\, dx, \qquad i, j = 1, \dots, m.$$

3. Use the Gram–Schmidt orthogonalization to construct the orthogonal polynomials Q_0, Q_1, and Q_2 on $[0,1]$ from $q_0(x) = 1$, $q_1(x) = x$, and $q_2(x) = x^2$.

4. Let $\{Q_n\}_{n=0}^{\infty}$ be orthonormal polynomials on $[a,b]$. Define $K_n(x,t) = \sum_{k=0}^{n} Q_k(x) Q_k(t)$ for all $n \geq 0$ and all $x, t \in \mathbb{R}$. Show that
$$p_n(x) = \int_a^b p_n(t) K_n(x,t)\, dt \qquad \forall p_n \in \mathcal{P}_n \text{ and } x \in \mathbb{R}.$$

5. Let $u, v \in C^2([0,1])$ be two nonzero functions such that $u(0) = u(1) = v(0) = v(1) = 0$. Let $\lambda, \mu \in \mathbb{R}$ be such that $\lambda \neq \mu$. Assume that $-u'' + u = \lambda u$ and $-v'' + v = \mu v$ on $[0,1]$. Prove that u and v are orthogonal in $L^2(0,1)$.

6. Let
$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(x^2 - 1 \right)^n \right], \qquad n = 0, 1, \dots$$
be the Legendre polynomials. Let $n \geq 1$. Prove directly by Rolle's Theorem that P_n has n simple roots in $(-1, 1)$.

7. Let
$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left[\left(x^2 - 1 \right)^n \right], \qquad n = 0, 1, \dots$$
be the Legendre polynomials. Let $r \geq 1$ be an integer. Show that
$$\int_{-1}^{1} P_m^{(r)}(x) P_n^{(r)}(x) (1 - x^2)^r dx = 0 \qquad \text{if } m, n \geq r \text{ and } m \neq n.$$