Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 7

Due: 1:00 pm, Wednesday, March 6, 2024.

1. Let $x_0, \dots, x_n \in [a,b]$ be distinct, $L_n : C([a,b]) \to \mathcal{P}_n$ the corresponding Lagrange interpolation operator, and $l_0, \dots, l_n$ the associated Lagrange basis polynomials. Prove the following:
   (1) $\|L_n f\|_{C([a,b])} \le \lambda_n \|f\|_{C([a,b])}$ for any $f \in C([a,b])$, where $\lambda_n = \max_{a \le x \le b} \sum_{j=0}^n |l_j(x)|$;
   (2) There exists a $f_0 \in C([a,b])$ such that $\|f_0\|_{C([a,b])} = 1$ and $\|L_n f_0\|_{C([a,b])} = \lambda_n \|f_0\|_{C([a,b])}$.

2. Let $n \ge 0$ be an integer and X_n the set of $(x_0, \dots, x_n) \in \mathbb{R}^{n+1}$ such that $x_0, \dots, x_n$ are distinct points in $[-1,1]$. Prove that the function $N(x_0, \dots, x_n) = \max_{-1 \le x \le 1} |(x - x_0) \dots (x - x_n)|$ has a unique minimizer in X_n. Find this minimizer and the minimum value.

3. Let $n \ge 1$ and $Q_0, \dots, Q_n$ be orthogonal polynomials on $[a,b]$. Let $x_1, \dots, x_n$ be the n distinct zeros of Q_n in $[a,b]$. Prove for any distinct points $y_1, \dots, y_n \in [a,b]$ that
$$\int_a^b (x - x_1)^2 \cdots (x - x_n)^2 dx \le \int_a^b (x - y_1)^2 \cdots (x - y_n)^2 dx.$$

4. Let $Q_0, Q_1, \dots$ be orthogonal polynomials on $[0,1]$. For each $n \ge 1$, let $x_1^{(n)}, \dots, x_n^{(n)}$ denote the n distinct zeros of Q_n in $[0,1]$ and $L_{n-1} : C([0,1]) \to \mathcal{P}_{n-1}$ the associated Lagrange interpolation operator. Let $f \in C([0,1])$. Prove that $\|f - L_{n-1} f\|_{L^2(0,1)} \to 0$.

5. Let $n = 1$ and $a = x_0 < x_1 = b$. Show that the Peano kernel $K_1(x,t)$ is given by
$$K_1(x,t) = \begin{cases} (t-a)(x-b)/(b-a) & \text{if } a \le t \le x \le b, \\ (x-a)(t-b)/(b-a) & \text{if } a \le x \le t \le b. \end{cases}$$
Let $h \in C([a,b])$ and define
$$u(x) = \int_a^b K_1(x,t) h(t)\, dt \qquad \forall x \in [a,b].$$
Show that $u''(x) = h(x)$ if $a < x < b$ and $u(a) = u(b) = 0$.

6. Let $x_1, \dots, x_n$ be n distinct points in $\mathbb{R}$ and $l_k \in \mathcal{P}_{n-1}$ $(k = 1, \dots, n)$ the associated Lagrange basis polynomials. Let $y_1, \dots, y_n; y_1', \dots, y_n' \in \mathbb{R}$. Define
$$\phi_k(x) = [1 - 2l_k'(x_k)(x - x_k)][l_k(x)]^2 \quad \text{and} \quad \psi_k(x) = (x - x_k)[l_k(x)]^2, \quad k = 1, \dots, n.$$
Prove that $\phi_k(x_j) = \delta_{kj}$, $\phi_k'(x_j) = 0$, $\psi_k(x_j) = 0$, and $\psi_k'(x_j) = \delta_{kj}$ for all $j, k = 1, \dots, n$.

7. Let $N \ge 1$ be an integer, $h = (b-a)/N$, and $x_j = a + jh$, $j = 0, \dots, N$. For any $f \in C([a,b])$, let $I_h f \in C([a,b])$ be such that $I_h f \in \mathcal{P}_1$ on each $[x_{j-1}, x_j]$ $(j = 1, \dots, N)$ and $I_h f(x_j) = f(x_j)$ $(j = 0, \dots, N)$. Prove the following:
   (1) If $f \in C^2([a,b])$ and $M_2 = \max_{a \le x \le b} |f''(x)|$, then
$$\max_{a \le x \le b} |f(x) - (I_h f)(x)| \le \frac{M_2}{8} h^2;$$
   (2) If $f \in C^3([a,b])$, $M_3 = \max_{a \le x \le b} |f'''(x)|$, and $m_j = (x_{j-1} + x_j)/2$ $(j = 1, \dots, N)$, then
$$\max_{1 \le j \le N} |f'(m_j) - (I_h f)'(m_j)| \le \frac{M_3}{24} h^2.$$