Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 8

Due: 1:00 pm, Friday, March 15, 2024.

1. Given a numerical integration formula on $[-1, 1]$

$$\int_{-1}^{1} g(t)\, dt \approx \sum_{j=1}^{n} a_j g(t_j). \tag{1}$$

Define for an interval $[a, b]$ that $A_j = (b-a)a_j/2$ and $x_j = [(b-a)t_j + a + b]/2$, $j = 1, \ldots, n$. Show that the numerical integration formula on $[a, b]$

$$\int_{a}^{b} f(x)\, dx \approx \sum_{j=1}^{n} A_j f(x_j)$$

has the same degree of precision as that of the formula (1).

2. Find A, B, C such that the weighted numerical quadrature

$$\int_{-2}^{2} |x| f(x)\, dx \approx Af(-1) + Bf(0) + Cf(1)$$

is exact for polynomials of degree as high as possible. Find the degree of precision of the quadrature.

3. Consider an interpolatory quadrature

$$\int_{a}^{b} f(x)\, dx \approx \sum_{k=0}^{n} A_k f(x_k).$$

Define for each integer $j \geq 0$

$$F_j(t) = \int_{a}^{b} (x-t)_+^j dx - \sum_{k=0}^{n} A_k (x_k - t)_+^j.$$

Show that

$$\int_{a}^{b} F_j(t) dt = 0, \qquad j = 0, \ldots, n-1.$$

4. Consider the trapezoidal formula

$$\int_{a}^{b} f(x)\, dx \approx \frac{1}{2}(b-a)\left[f(a) + f(b)\right].$$

(1) Show that the degree of precision of the formula is $m = 1$.
(2) Calculate the Peano kernel K_1 of the formula and show that it does not change sign in $[a, b]$.
(3) Let $f \in C^2([a, b])$. Show that there exists $\xi \in (a, b)$ such that

$$\int_{a}^{b} f(x)\, dx - \frac{1}{2}(b-a)\left[f(a) + f(b)\right] = -\frac{1}{12}(b-a)^3 f''(\xi).$$

(4) Let $N \geq 1$ be an integer, $h = (b-a)/N$, and $x_j = a + jh$, $j = 0, \ldots, N$. Let $f \in C^2([a,b])$. Prove that there exists $\eta \in (a,b)$ such that

$$\int_a^b f(x)\,dx - \left\{ \frac{h}{2}\,[f(a)+f(b)] + h \sum_{j=1}^{N-1} f(x_j) \right\} = -\frac{(b-a)f''(\eta)}{12} h^2,$$

5. Consider the Newton–Cotes formula

$$\int_a^b f(x)\,dx \approx \sum_{j=0}^{n} A_j f(x_j)$$

with $n+1$ points $x_j = a + j(b-a)/n$, $j = 0, \ldots, n$.

(1) Show that $A_j = A_{n-j}$ for $j = 0, \ldots, [n/2]$.

(2) Show that the degree of precision of the formula is n if n is odd and is $n+1$ if n is even.

6. Let $p_3 \in \mathcal{P}_3$ be the Hermite interpolation polynomial of $f \in C^1([a,b])$ determined by

$$p_3(a) = f(a), \quad p_3'(a) = f'(a), \quad p_3(b) = f(b), \quad p_3'(b) = f'(b).$$

(1) Show that

$$\int_a^b p_3(x)\,dx = \frac{1}{2}(b-a)[f(a)+f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)].$$

(2) Determine the degree of precision of the numerical quadrature

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)[f(a)+f(b)] - \frac{1}{12}(b-a)^2[f'(b) - f'(a)].$$