Math 270B: Numerical Analysis (Part B)
Winter quarter 2024

Homework Assignment 9

Due: 5:00 pm, Wednesday, March 20, 2024.

1. Let $f : [a, b] \to \mathbb{R}$ be integrable over $[a, b]$. Let $n \geq 1$ be an integer, $h = (b - a)/(2n)$, and $x_i = a + ih$ $(i = 0, \ldots, 2n)$.
   (1) Derive the composite Simpson's formula for the integration of a function of f over $[a, b]$ by applying the basic Simpson's formula to each of the subinterval $[x_{2i-2}, x_{2i}]$ $(i = 0, \ldots, n)$.
   (2) Assume $f \in C^4([a, b])$. Derive an error formula for the composite Simpson's formula that is derived in Part (1).

2. Let $\{Q_n\}_{n=0}^{\infty}$ be a system of orthogonal polynomials on $[a, b]$. Fix $n \geq 1$. Let $x_1, \ldots, x_n$ be the n distinct roots of Q_n in (a, b). Let
$$\int_a^b f(x)\,dx \approx \sum_{j=1}^n A_j f(x_j)$$
be the corresponding Gaussian quadrature. Show that $\sum_{j=1}^n A_j Q_k(x_j) = 0$ $(k = 1, \ldots, 2n-1)$.

3. Consider a Gaussian formula
$$\int_a^b f(x)\,dx \approx \sum_{j=1}^n A_j f(x_j).$$
Show that for any $f \in C([a, b])$ the error
$$e_n(f) = \int_a^b f(x)\,dx - \sum_{j=1}^n A_j f(x_j)$$
satisfies
$$|e_n(f)| \leq 2(b - a) \min_{q \in \mathcal{P}_{2n-1}} \|f - q\|_{C([a,b])}.$$

4. Let $n \geq 1$ be an integer. The Gauss–Chebyshev quadrature is the weighted Gaussian quadrature on $[-1, 1]$ with the weight $1/\sqrt{1 - x^2}$ using $x_j = \cos((2j - 1)\pi/2n)$ $(j = 1, \ldots, n)$, the n roots of the nth Chebyshev polynomial $T_n(x) = \cos(n \arccos x)$. Show that the Gauss–Chebyshev formula is given by
$$\int_{-1}^1 \frac{f(x)}{\sqrt{1 - x^2}} dx \approx \frac{\pi}{n} \sum_{j=1}^n f(x_j).$$

5. Let $f \in C([a, b])$ and denote by $I(f)$ the integral of f over $[a, b]$. Let $N \geq 1$ be an integer, $h = (b - a)/2N$, and $x_j = a + jh$, $j = 0, \ldots, 2N$. Let T_N, T_{2N}, and S_N denote, respectively, the approximate value of $I(f)$ by the composite trapezoidal rule with N subintervals $[x_{2j-2}, x_{2j}]$, $j = 1, \ldots, N$, by the composite trapezoidal rule with $2N$ subintervals $[x_{j-1}, x_j]$, $j = 1, \ldots, 2N$, and by the composite Simpson rule with N subintervals $[x_{2j-2}, x_{2j}]$, $j = 1, \ldots, N$. Prove that $S_N = (4T_{2N} - T_N)/3$.