140C Office Hours

Why is $\|A^{-1}\|$ used in Thm 9.8?

$\|A\|$ only bounds how much A can lengthen vectors.
$\frac{1}{\|A^{-1}\|}$ bounds how much A shrinks vectors (and the concern for invertibility is that a nonzero vector gets shrunk to the 0 vector)

Suppose $A \in L(\mathbb{R}^n)$ is invertible

for all $y \in \mathbb{R}^n$ $\quad |A^{-1}y| \leq \|A^{-1}\| \cdot |y|$

setting $x = A^{-1}y$ we obtain

$$|x| \leq \|A^{-1}\| \cdot |Ax|$$

Meaning $\quad |Ax| \geq \frac{1}{\|A^{-1}\|}|x|$

This holds for all $x \in \mathbb{R}^n$

In general, (still assuming A invertible)

$(*)$ $\forall x \in \mathbb{R}^n \quad \frac{1}{\|A^{-1}\|} \cdot |x| \leq |Ax| \leq \|A\| \cdot |x|$

and $\|A\|$ and $\|A^{-1}\|$ are the smallest real numbers for which the above statement $(*)$ is true

In Thm 9.8 require $\|B-A\| < \frac{1}{\|A^{-1}\|}$

Why are the two formulas for $\|A\|$ equal?

By definition $\quad \|A\| = \sup\limits_{\substack{x \in \mathbb{R}^n \\ |x| \leq 1}} |Ax|$

Clearly $\|A\| \geq \sup\limits_{x \in \mathbb{R}^n, |x|=1} |Ax|$ since $\{x \in \mathbb{R}^n : |x| = 1\} \subseteq \{x \in \mathbb{R}^n : |x| \leq 1\}$.

On the other hand, consider any $x \in \mathbb{R}^n$ with $|x| \leq 1$.

<u>Case 1</u>: $x = 0$. Then $Ax = 0$ so $|Ax| = 0 \leq \sup\limits_{y \in \mathbb{R}^n, |y|=1} |Ay|$.

<u>Case 2</u>: $x \neq 0$. Set $t = \frac{1}{|x|}$. Then $|tx| = 1$ and since $t \geq 1$ we have

$$|Ax| \leq t|Ax| = |Atx| \leq \sup_{y \in \mathbb{R}^n, |y|=1} |Ay|$$

So $\sup\limits_{y \in \mathbb{R}^n, |y|=1} |Ay|$ is an upper bound to $\{|Ax| : x \in \mathbb{R}^n, |x| \leq 1\}$.

Therefore $\|A\| \leq \sup\limits_{y \in \mathbb{R}^n, |y|=1} |Ay|$. We conclude

$$\|A\| = \sup_{y \in \mathbb{R}^n, |y|=1} |Ay|$$

Why is $GL(\mathbb{R}^n)$ open?

Recall if (X, d) metric space and $U \subseteq X$, then
U is <u>open</u> if $\forall x \in U \ \exists r > 0 \ B_r(x) \subseteq U$.

$GL(\mathbb{R}^n)$ is open by Theorem 9.8 A since
for every $A \in GL(\mathbb{R}^n)$ $\underbrace{B_{\frac{1}{\|A^{-1}\|}}(A) \subseteq GL(\mathbb{R}^n)}$

$\forall B \in L(\mathbb{R}^n)$

$$B \in B_{\frac{1}{\|A^{-1}\|}}(A) \Rightarrow \|B - A\| < \frac{1}{\|A^{-1}\|} \xRightarrow{\text{Thm } 9.8 \text{ A}} B \in GL(\mathbb{R}^n)$$

Ch. 9 #8

$f: E \to \mathbb{R}$, $E \subseteq \mathbb{R}^n$

f differentiable, local max at x. Show $f'(x) = 0$.

Write $f(x+h) - f(x) = f'(x)h + r(h)$ where $\frac{|r(h)|}{|h|} \to 0$ as $h \to 0$

Towards a contradiction, suppose $f'(x) \neq 0$.
Then there is $h \in \mathbb{R}^n$ with $f'(x)h \neq 0$.
By replacing h with $-h$ if necessary, can assume $f'(x)h > 0$.

Set $\varepsilon = \frac{1}{2|h|} f'(x)h$. Then for all $t > 0$ close enough to 0 so that $\frac{|r(th)|}{|th|} < \varepsilon$ we have

$$
\begin{aligned}
f(x+th) - f(x) &= t f'(x)h + r(th) \\
&\geq t f'(x)h - |r(th)| \\
&\geq t f'(x)h - \varepsilon t |h| \\
&= t(f'(x)h - \varepsilon |h|) \\
&\geq \tfrac{1}{2} t f'(x)h > 0.
\end{aligned}
$$

Thus x is not a local max, contradiction.

If $g(x) = Af(x)$ why is $g'(x) = Af'(x)$?

Claim: If $A \in L(\mathbb{R}^m, \mathbb{R}^k)$, $f: \mathbb{R}^n \to \mathbb{R}^m$ and $g(x) = Af(x)$ then $g'(x) = Af'(x)$ (assuming $f'(x)$ exists)

Pf: We have

$$\lim_{h \to 0} \frac{|g(x+h) - g(x) - Af'(x)h|}{|h|}$$

$$= \lim_{h \to 0} \frac{|A(f(x+h) - f(x) - f'(x)h)|}{|h|}$$

$$\leq \lim_{h \to 0} \|A\| \cdot \frac{|f(x+h) - f(x) - f'(x)h|}{|h|} = 0 \quad \square$$

Why do Lebesgue-measurable non-Borel measurable sets exist?

Fact: Let m be Lebesgue measure on $\mathbb{R}^p$. If $A \in \mathcal{M}(m)$ and $m(A)=0$ then $B \in \mathcal{M}(m)$ for all $B \subseteq A$.

Pf: Since $0 = m(A) = m^*(A)$ and m^* is monotone, so $\forall B \subseteq A$ $m^*(B)=0$. This means $B \in \mathcal{M}_F(m) \subseteq \mathcal{M}(m)$ because $B_n \to B$ where $B_n = \emptyset \in \mathcal{E}$
($B_n \to B$ since $\lim_{n\to\infty} m^*(B \triangle B_n) = m^*(B) = 0$)

Fact: If C is the Cantor set then $m(C)=0$.

Pf: Given in class.

So C is an uncountable compact set and every subset of C is Lebesgue measurable.

However, the collection of Borel sets contained in C coincides with the Borel σ-algebra of C
(For any metric space (X,d) the Borel σ-algebra is by definition the smallest σ-algebra containing all open sets)

Fact: If (X,d) is an uncountable separable (has a cntbl dense subset) complete metric space, then there exist subsets of X that are not Borel

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Fact: (X,d) uncntbl, separable, complete.
Then $|\mathcal{B}(X)| = |\mathbb{R}|$.

Pf: Claim: $|\{U \subseteq X : U \text{ open}\}| = |\mathbb{R}|$

Pf: Let $X_0 \subseteq X$ be cntbl dense.
Set $\Upsilon = \{B_q(x) : q \geq 0, q \in \mathbb{Q}, x \in X_0\}$
For any open $U \subseteq X$ and any $x \in U$
there is $B_q(x') \in \Upsilon$ with $x \in B_q(x') \subseteq U$.
Enumerate Υ as $V_0, V_1, \ldots$
This shows $\forall$ open $U \subseteq X$ $\exists I \subseteq \mathbb{N}$ $U = \bigcup_{i \in I} V_i$
Therefore $|\{U \subseteq X : U \text{ open}\}| \leq |\{I : I \subseteq \mathbb{N}\}| = |\mathbb{R}|$
Reverse inequality $\geq$ ....?

Clearly true for Cantor set by considering the open sets $(-\infty, x) \cap C$ for $x \in C$

Claim implies $|\mathcal{B}(X)| \geq |\mathbb{R}|$.
Let $\mathcal{N}$ = Baire space = $\{x : \mathbb{N} \to \mathbb{N}\}$

Fact: Every Borel set $A \subseteq X$ is the projection of a closed set $B \subseteq X \times \mathcal{N}$
$(A = \{x \in X : \exists y \in \mathcal{N} \ (x,y) \in B\})$

By first claim, $X \times \mathcal{N}$ has at most $|\mathbb{R}|$-many closed sets. So by above fact $|\mathcal{B}(X)| \leq |\mathbb{R}|$. $\square$

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Fact: $|\{Y : Y \subseteq X\}| > |X| = |\mathbb{R}|$

Pf: "$>$" by Math 109

$\mathcal{N}$ is a metric space: if $x, y : \mathbb{N} \to \mathbb{N}$
$d(x,y) = \inf\{2^{-n} : n \in \mathbb{N} \ \forall k < n \ x(k) = y(k)\} \leq 1$

Fact: Every Borel set $A \subseteq X$ is the projection of a closed set $B \subseteq X \times \mathcal{N}$
$(A = \{x \in X : \exists y \in \mathcal{N} \ (x,y) \in B\})$

Pf Sketch: Set $\mathcal{A} = \{A \subseteq X : \exists \text{ closed } B \subseteq X \times \mathcal{N} \text{ with } \pi_X(B) = A\}$.
Check that $\mathcal{A}$ is a σ-algebra and contains every closed subset of X. Since $\mathcal{B}(X)$ is the smallest σ-algebra containing all closed sets, $\mathcal{B}(X) \subseteq \mathcal{A}$

Measurable Functions

Fact: $f: X \to [-\infty, +\infty]$ is measurable iff
$$\forall A \in \mathcal{B}(\mathbb{R}) \quad f^{-1}(A) \in \mathcal{M}$$

In general, if $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are measure spaces then a function $f: X \to Y$ is measurable if
$$\forall A \in \mathcal{N} \quad f^{-1}(A) \in \mathcal{M}$$

So for the definition of measurable we use in class, we take the codomain $\mathbb{R}$ to be equipped with $\mathcal{B}(\mathbb{R})$ by default.

$$x \in X \longmapsto f(x) \in \mathbb{R} \longmapsto g(f(x)) \in \mathbb{R}$$

Chapter 11 Problem 12

The answer is yes. To see this, it suffices to show that whenever a sequence (x_n) in $[0,1]$ converges to x, $g(x_n) \to g(x)$ as $n \to \infty$.

Consider such a seq. (x_n). Define $f_n(y) = f(x_n, y)$. By (b) $\forall y$ $f_n(y) \to f(x,y)$. Also $\forall n \forall y$ $|f_n(y)| \leq 1$ and $\int_0^1 1\, dy < \infty$, so by Lebesgue Dominated Convergence Theorem

$$g(x) = \int_0^1 f(x,y)\,dy = \int_0^1 \lim_{n\to\infty} f_n(y)\,dy$$

$$= \lim_{n\to\infty} \int_0^1 f_n(y)\,dy = \lim_{n\to\infty} g(x_n). \qquad \square$$

Fatou's Theorem – Different Perspective

Fix $\varepsilon > 0$.

Define $E_n = \{ x \in E : \inf_{m \geq n} f_m(x) > (1-\varepsilon) f(x) \}$

Then $E_1 \subseteq E_2 \subseteq \cdots$ and since $f(x) = \liminf_{n\to\infty} f_n(x)$ (*)

we have $\forall x \in E \ \exists n \ x \in E_n$, or equivalently

$\bigcup_{n=1}^{\infty} E_n = E$.

Since the f_n's are non-negative, we have

$$\forall m \geq n \qquad \int_{E_n} (1-\varepsilon) f \, d\mu \leq \int_{E_n} f_m \, d\mu \leq \int_{E_n} f_m \, d\mu + \int_{E \setminus E_n} f_m \, d\mu$$

$$= \int_E f_m \, d\mu$$

Taking liminf as $m \to \infty$ we obtain

$$\int_{E_n} (1-\varepsilon) f \, d\mu \leq \liminf_{m\to\infty} \int_E f_m \, d\mu$$

Next by Theorem 11.24 and Theorem 11.3 we can take limit as $n \to \infty$ to obtain

$$(1-\varepsilon)\int_E f \, d\mu = \int_E (1-\varepsilon) f \, d\mu = \lim_{n\to\infty} \int_{E_n} (1-\varepsilon) f \, d\mu \leq \liminf_{m\to\infty} \int_E f_m \, d\mu$$

Now take limit as $\varepsilon \to 0$.

(*) $f(x) = \liminf_{m\to\infty} f_m(x) = \lim_{n\to\infty} \inf_{m \geq n} f_m(x)$

Summary: If n is large enough, for "most" points $x \in E$ we have $(1-\varepsilon) f(x) < f_n(x)$ and thus

$$\int_E (1-\varepsilon) f \, d\mu \lessapprox \int_E f_n \, d\mu$$

Chapter 11 Problem 7

Modified Theorem 11.33:

Let $a \le b$ be real numbers, let $\alpha : \mathbb{R} \to \mathbb{R}$ be monotone increasing. Define

$$\alpha_1(x) = \begin{cases} \alpha(a) & \text{if } x < a \\ \alpha(x) & \text{if } a \le x \le b \\ \alpha(b) & \text{if } x > b. \end{cases}$$

Let μ be the measure obtained from using α_1 in Ex. 6 (b).

Ⓐ If $f \in \mathcal{R}_a^b(\alpha)$ then $f \in L([a,b], \mu)$ and

$$\int_a^b f \, d\mu = \mathcal{R}\int_a^b f \, d\alpha$$

Ⓑ Suppose f is bounded. Then $f \in \mathcal{R}_a^b(\alpha)$ iff f is left-cont. at every point α_1 is not left-cont, f is right-cont. at every point α_1 is not right-cont, and the set of points where α_1 is continuous and f is discontinuous has μ-measure 0.