Math 140A Winter 2020
Second Midterm

Instructions: Write your proofs clearly and legibly in complete sentences. Unless otherwise indicated, you must show all work. You may freely use any results we learned in class.
1. (10 points) Let \((X,d)\) be a metric space and let \(K \subseteq X\) be compact. Suppose that \(E \subseteq K\) and \(E' = \emptyset\). Prove that \(E\) is finite.

(This is the contrapositive of a theorem in our textbook. You may use any results we learned in class aside from that theorem.)

**Solution:** For every \(x \in K\) we have \(x \not\in E'\), meaning there is \(r_x > 0\) so that if we set \(V_x = B_{r_x}(x)\) then \((V_x \setminus \{x\}) \cap E = \emptyset\) (or equivalently \(V_x \cap E \subseteq \{x\}\)). The collection \(\{V_x : x \in K\}\) is an open cover of \(K\), so there are \(x_1, x_2, \ldots, x_n \in K\) with \(K \subseteq \bigcup_{i=1}^n V_{x_i}\). Since \(E \subseteq K\) we have

\[
E = E \cap K \subseteq E \cap \left(\bigcup_{i=1}^n V_{x_i}\right) = \bigcup_{i=1}^n (E \cap V_{x_i}) \subseteq \bigcup_{i=1}^n \{x_i\} = \{x_1, x_2, \ldots, x_n\}.
\]

We conclude \(|E| \leq n\). \(\Box\)
2. (11 points) Let \((p_n)\) be a sequence in a metric space \((X, d)\).

(a) (4 points) State the definition of what it means for \((p_n)\) to be Cauchy.

(b) (7 points) Prove that if \((X, d)\) is complete and \(d(p_n, p_{n+1}) \leq 2^{-n-1}\) for all \(n \geq 1\) then \((p_n)\) converges.

Solution: (a). \((p_n)\) is Cauchy if for every \(\epsilon > 0\) there is \(N \in \mathbb{N}\) so that for all \(m \geq n \geq N\) we have \(d(p_n, p_m) < \epsilon\).

(b). Let \(\epsilon > 0\) and pick \(N \in \mathbb{N}\) large enough that \(2^{-N} < \epsilon\). If \(m \geq n \geq N\) then

\[
d(p_n, p_m) \leq d(p_n, p_{n+1}) + d(p_{n+1}, p_{n+2}) + \ldots + d(p_{m-1}, p_m) \\
\leq 2^{-n-1} + 2^{-n-1} + \ldots + 2^{-m} \\
= \frac{2^{-n-1} - 2^{-m-1}}{1 - 2^{-1}} \\
= 2^{-n} - 2^{-m} \\
\leq 2^{-n} \\
\leq 2^{-N} \\
< \epsilon.
\]

Thus \((p_n)\) is Cauchy. Since \((X, d)\) is complete, \((p_n)\) converges.
3. (10 points) Set \( a_1 = 1 \) and \( a_{n+1} = \frac{a_n}{3} + 2 \) for \( n \geq 1 \). Prove that \( \lim a_n = 3 \). (Hint: First show the sequence is monotone.)

**Solution:** Notice that \( a_1 < 3 \) and that if \( a_n < 3 \) then \( a_{n+1} = \frac{a_n}{3} + 2 < \frac{3}{3} + 2 = 3 \). So by induction \( a_n < 3 \) for all \( n \geq 1 \). Since \( a_n < 3 \) it follows that \( a_n - \frac{a_n}{3} = \frac{2}{3}a_n < 2 \) and hence \( a_n < \frac{2a_n}{3} + 2 = a_{n+1} \). Therefore \((a_n)\) is monotone increasing and bounded, so \( a = \lim a_n \) exists. Finally, we have

\[
a = \lim a_n = \lim a_{n+1} = \lim \left( \frac{a_n}{3} + 2 \right) = \frac{\lim a_n}{3} + 2 = \frac{a}{3} + 2.
\]

Solving for \( a \), we obtain \( a = 3 \). \( \square \)
4. (9 points) Determine if the following series converge or diverge. You may use any of the convergence/divergence results we learned in class, but you must show justifying work.

(a) (3 points) $\sum (-1)^n \frac{2n+1}{3n+2}$

Solution: $\lim_{n \to \infty} \frac{2n+1}{3n+2} = \lim_{n \to \infty} \frac{2+\frac{1}{n}}{3+\frac{2}{n}} = \frac{2}{3}$. Since the terms do not converge to 0, the series diverges.

(b) (3 points) $\sum \frac{(n!)^2}{(2n+1)!}$

Solution: $\frac{(n!)^2}{(2n)!} = \frac{1 \cdot 2 \cdot \ldots \cdot n}{(n+1) \cdot (n+2) \cdot \ldots \cdot (n+n)} \leq \frac{1}{2} \cdot \frac{1}{2} \cdots \frac{1}{2} = \left(\frac{1}{2}\right)^n$. The series converges via comparison with the geometric series $\sum \left(\frac{1}{2}\right)^n$. (Alternatively, one could apply the ratio test.)

(c) (3 points) $\sum \frac{1}{(\sqrt{n+1}+\sqrt{n})^3}$

Solution: $\frac{1}{(\sqrt{n+1}+\sqrt{n})^3} < \frac{1}{(2\sqrt{n})^3} = \frac{1}{8} \cdot \frac{1}{n^{3/2}}$. The series converges via comparison with $\sum \frac{1}{8} \cdot \frac{1}{n^{3/2}}$. 
