Instructions:

• (Formatting) Use your own sheets of paper. You do not need to rewrite the questions - just write the problem number.
• (Allowed Resources) The exam is open book. Additionally, you may consult your lecture notes, homework assignments, the first two midterms, and the practice problems to the exams.
• (Prohibited Resources) You are not allowed to seek help from other human beings, online resources, or any other resources not affiliated with our class. If I or your TA find anything suspicious about your solutions or your exam performance, you may be required to have a video chat with me where I may ask you about your solutions or ask you how to solve similar problems. Any evidence of cheating will be taken seriously and may have severe consequences.
• (Solutions) Your solutions should be written clearly, in complete sentences, and show all justifying work. In your solutions you can apply theorems that we learned in class (unless the problem says otherwise), but you cannot apply results from exercises, homework, or past exams (those materials are for reference only).
• (Questions) If you have any questions about the exam, I will be standing by - just email me at bseward@ucsd.edu.
• (Finishing) The exam ends at 2:30 pm. After finishing the exam, you must scan or take photos of your solutions and upload them to Gradescope. You are allowed 10 minutes to upload your solutions - the deadline for submission on Gradescope is 2:40 pm. If you have any technical difficulties uploading your work to Gradescope, you must email me the images or pdf of your solutions by 2:40.

1. (7 points) Let $E \subseteq \mathbb{R}$ be uncountable and fix a real number $x_0 \in \mathbb{R}$. Prove there is $\delta > 0$ so that $E \setminus (x_0 - \delta, x_0 + \delta)$ is uncountable.

2. (9 points) Let $(X,d)$ be a metric space and let $f : X \to \mathbb{C}$ be a continuous function. Prove directly from the $\epsilon\-\delta$ definition of continuity that the set $E = \{ x \in X : |f(x)| > 1 \}$ is open. (You can not use any of the theorems we learned about continuous functions).

3. (10 points) Let $(X,d_X)$ and $(Y,d_Y)$ be metric spaces, let $E \subseteq X$, and let $f : E \to Y$. Assume that $E$ is compact and perfect and that $f : E \to Y$ is continuous and injective. Prove that $f(E)$ is perfect.
(Hint: Don’t get distracted by the assumptions; focus on what you want to prove. One by one, the assumptions will become useful.)

4. (8 points) Let $(X,d)$ be a metric space, let $(p_n)$ be a sequence in $X$, and set $E = \{ p_n : n \in \mathbb{N} \}$. Prove that if $E$ is not closed then $(p_n)$ admits a subsequence $(p_{n_k})$ that is convergent.

5. (10 points) Let $(a_n)$ and $(b_n)$ be bounded sequences of real numbers. Suppose that $a = \lim a_n$ exists. Prove that $\lim sup(a_n + b_n) = a + \lim sup b_n$.

6. (7 points) Let $(a_n)$ be a sequence of non-zero complex numbers and suppose that the power series $\sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $\frac{1}{2}$. Prove that there is a subsequence $(a_{n_k})$ so that $\sum_{k=0}^{\infty} \frac{1}{a_{n_k}}$ converges absolutely.

7. (9 points) Let $(X,d)$ be a metric space, let $E \subseteq X$, and let $f : E \to \mathbb{R}$. Suppose there is a point $p \in E' \setminus E$ and a sequence $(p_n)$ in $E$ converging to $p$ with $\lim inf f(p_n) = -3$ and $\lim sup f(p_n) = 3$. Prove that $f$ is not uniformly continuous.