Final Exam - Math 140A Winter 2020

Instructions:
1. (7 points) Let \( E \subseteq \mathbb{R} \) be uncountable and fix a real number \( x_0 \in \mathbb{R} \). Prove there is \( \delta > 0 \) so that \( E \setminus (x_0 - \delta, x_0 + \delta) \) is uncountable.

Solution: We will prove the contrapositive. So suppose that \( E \setminus (x_0 - \delta, x_0 + \delta) \) is countable for every \( \delta > 0 \). In particular, for each \( n \in \mathbb{Z}_+ \), the set \( E_n = E \setminus (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \) is countable. Since a countable union of countable sets is countable, we have that \( \bigcup_{n \in \mathbb{Z}_+} E_n \) is countable. It follows that \( E \) is countable since \( E \setminus \{x_0\} = \bigcup_{n \in \mathbb{Z}_+} E_n \).

2. (9 points) Let \((X, d)\) be a metric space and let \( f : X \to \mathbb{C} \) be a continuous function. Prove directly from the \( \epsilon-\delta \) definition of continuity that the set \( E = \{x \in X : |f(x)| \geq 1\} \) is open. (You can not use any of the theorems we learned about continuous functions).

Solution: Let \( x_0 \in E \) and set \( \epsilon = |f(x)| - 1 > 0 \). Since \( f \) is continuous, there is \( \delta > 0 \) so that whenever \( x \in X \) satisfies \( d(x, x_0) < \delta \) we have \( |f(x) - f(x_0)| < \epsilon \). We claim that \( B_\delta(x_0) \subseteq E \). Indeed, if \( x \in B_\delta(x_0) \) then \( d(x, x_0) < \delta \) so

\[
|f(x)| \geq |f(x_0)| - |f(x) - f(x_0)| > |f(x_0)| - \epsilon = 1
\]

and thus \( x \in E \). We conclude that \( E \) is open.

3. (10 points) Let \((X, d_X)\) and \((Y, d_Y)\) be metric spaces, let \( E \subseteq X \), and let \( f : E \to Y \). Assume that \( E \) is compact and perfect and that \( f : E \to Y \) is continuous and injective. Prove that \( f(E) \) is perfect.

(Hint: Don’t get distracted by the assumptions; focus on what you want to prove. One by one, the assumptions will become useful.)

Solution: Since \( E \) is compact and \( f \) is continuous, \( f(E) \) is compact and therefore closed, meaning \( f(E)' \subseteq f(E) \). To show that \( f(E) \) is perfect, it only remains to show that \( f(E) \subseteq f(E)' \). So let \( y \in f(E) \) and let \( r > 0 \). Pick any \( x \in E \) with \( f(x) = y \). Since \( f \) is continuous, there is \( \delta > 0 \) so that whenever \( p \in E \) satisfies \( d_X(p, x) < \delta \) we have \( d_Y(f(p), y) = d_Y(f(p), f(x)) < r \). Since \( E \) is perfect and \( x \in E \), we have \( x \in E' \) and thus \( (B_\delta(x) \setminus \{x\}) \cap E \neq \emptyset \). Fix any point \( p \in (B_\delta(x) \setminus \{x\}) \). Then \( d_X(p, x) < \delta \) so \( d_Y(f(p), y) < r \). Moreover, \( p \neq x \) and since \( f \) is injective we obtain \( f(p) \neq f(x) = y \). Therefore \( f(p) \in (B_r(y) \setminus \{y\}) \cap f(E) \). This shows that \( y \in f(E)' \). We conclude that \( f(E) \) is perfect.
4. (8 points) Let \((X,d)\) be a metric space, let \((p_n)\) be a sequence in \(X\), and set \(E = \{p_n : n \in \mathbb{N}\}\). Prove that if \(E\) is not closed then \((p_n)\) admits a subsequence \((p_{n_k})\) that is convergent.

Solution: Since \(E\) is not closed, \(E' \setminus E \neq \emptyset\). Fix any point \(p \in E' \setminus E\). Set \(n_1 = 1\). Proceeding by induction, suppose that \(n_1, \ldots, n_{k-1}\) have been defined. Since \(p \in E'\), we have that \(B_{1/k}(p) \cap E\) is infinite. So we can pick \(n_k > n_{k-1}\) so that \(p_{n_k} \in B_{1/k}(p)\). This defines the subsequence \((p_{n_k})\).

We claim that \((p_{n_k})\) converges to \(p\). Indeed, let \(\epsilon > 0\) and pick an integer \(K > 1/\epsilon\). Then for all \(k \geq K\) we have \(p_{n_k} \in B_{1/k}(p)\) so \(d(p_{n_k}, p) < 1/k \leq 1/K < \epsilon\). We conclude \((p_{n_k})\) converges to \(p\).

\(\square\)

5. (10 points) Let \((a_n)\) and \((b_n)\) be bounded sequences of real numbers. Suppose that \(a = \lim a_n\) exists. Prove that \(\lim \sup (a_n + b_n) = a + \lim \sup b_n\).

Solution: Since \(\lim \sup b_n\) is a subsequential limit of \((b_n)\), we can pick a subsequence \((b_{n_k})\) with \(\lim b_{n_k} = \lim \sup b_n\). Since \(a_n\) converges to \(a\) we have that \(a_{n_k}\) converges to \(a\) as well. Therefore

\[
\lim (a_{n_k} + b_{n_k}) = (\lim a_{n_k}) + (\lim b_{n_k}) = a + \lim \sup b_n.
\]

This shows that \(a + \lim \sup b_n\) is a subsequential limit of \((a_n + b_n)\). To finish the proof, we need to show that this is the largest subsequential limit of \((a_n + b_n)\).

Let \(\epsilon > 0\). Since \(a_n \to a\) we can pick \(N_1\) such that \(\forall n \geq N_1\) \(a_n < a + \frac{\epsilon}{2}\). By Theorem 3.17(b) we can pick \(N_2\) such that \(\forall m \geq N_2\) \(b_n < (\lim \sup b_n) + \frac{\epsilon}{2}\). It follows that for all \(m \geq \max(N_1, N_2)\) we have \(a_{n_k} + b_{n_k} < a + (\lim \sup b_n) + \epsilon\). Consequently, every subsequential limit of \((a_n + b_n)\) is at most \(a + (\lim \sup b_n) + \epsilon\). This holds for every \(\epsilon > 0\), so we find that every subsequential limit of \((a_n + b_n)\) is at most \(a + \lim \sup b_n\). We conclude \(\lim \sup (a_n + b_n) = a + \lim \sup b_n\).

\(\square\)

6. (7 points) Let \((a_n)\) be a sequence of non-zero complex numbers and suppose that the power series \(\sum_{n=0}^{\infty} a_n z^n\) has radius of convergence \(\frac{1}{2}\). Prove that there is a subsequence \((a_{n_k})\) so that \(\sum_{k=0}^{\infty} \frac{1}{a_{n_k}}\) converges absolutely.

Solution: The radius of convergence of \(\sum a_n z^n\) being \(\frac{1}{2}\) means that \(\lim \sup \sqrt[n]{|a_n|} = 2\). Pick a subsequence \((a_{n_k})\) so that \(|a_{n_k}| \geq 1\) and \(\sqrt[n]{|a_{n_k}|} \) converges to \(2\). Then

\[
\lim_{k \to \infty} \sqrt[n]{\frac{1}{a_{n_k}}} = \lim_{k \to \infty} \sqrt[n]{\frac{1}{|a_{n_k}|}} = \frac{1}{\lim_{k \to \infty} \sqrt[n]{|a_{n_k}|}} = \frac{1}{2}.
\]

Since \(|a_{n_k}| \geq 1\) and \(k \leq n_k\) we have \(\sqrt[n]{\frac{1}{a_{n_k}}} \leq \sqrt[n]{\frac{1}{a_{n_k}}}.\) Therefore

\[
\lim \sup_{k \to \infty} \sqrt[n]{\frac{1}{a_{n_k}}} \leq \lim \sup_{k \to \infty} \sqrt[n]{\frac{1}{a_{n_k}}} = \lim_{k \to \infty} \sqrt[n]{\frac{1}{a_{n_k}}} = \frac{1}{2} < 1,
\]

so \(\sum_{k=0}^{\infty} \frac{1}{a_{n_k}}\) converges by the root test and thus \(\sum_{k=0}^{\infty} \frac{1}{a_{n_k}}\) converges absolutely.

\(\square\)

7. (9 points) Let \((X,d)\) be a metric space, let \(E \subseteq X\), and let \(f : E \to \mathbb{R}\). Suppose there is a point \(p \in E' \setminus E\) and a sequence \((p_n)\) in \(E\) converging to \(p\) with \(\lim \inf f(p_n) = -3\) and \(\lim \sup f(p_n) = 3\). Prove that \(f\) is not uniformly continuous.

Solution: Set \(\epsilon = 4\) and let \(\delta > 0\). Since \(p_n \to p\), we can pick \(N\) such that \(\forall n \geq N\) \(d(p_n, p) < \delta\). Since a subsequential limit of \((f(p_n))\) converges to \(\lim \inf f(p_n) = -3\), we can find \(k \geq N\) with
$f(p_k) < -2$. Similarly, since a subsequence of $(f(p_n))$ converges to $\lim\sup f(p_n) = 3$, we can find $m \geq N$ with $f(p_m) > 2$. Then $d(p_k, p_m) \leq d(p_k, p) + d(p, p_m) < \delta$ but $|f(p_m) - f(p_k)| > 4 = \epsilon$. Since $\delta > 0$ was arbitrary, we conclude that $f$ is not uniformly continuous.  $\square$