1. Let $(X, d)$ be a metric space and let $K \subseteq X$ be compact. Prove that $K$ is closed.

**Solution:** We will show that $K^c$ is open. So fix $p \in K^c$. We must check that $p$ is an interior point of $K^c$.

For each $x \in K$ set $V_x = B_{\frac{1}{3}d(x,p)}(x)$. Notice that $d(x, p) > 0$ and thus $x \in V_x$. So $\{V_x : x \in K\}$ is an open cover of $K$. Since $K$ is compact, there are $x_1, x_2, \ldots, x_n \in K$ with $K \subseteq \bigcup_{i=1}^n V_{x_i}$. Set $r = \frac{1}{3} \min\{d(x_i, p) : 1 \leq i \leq n\}$ and notice that $r > 0$. For each $1 \leq i \leq n$ we have

$$V_{x_i} \cap B_r(p) \subseteq B_{\frac{1}{3}d(x_i,p)}(x_i) \cap B_{\frac{1}{3}d(x_i,p)}(p) = \emptyset.$$ 

Therefore

$$K \cap B_r(p) \subseteq \bigcup_{i=1}^n (V_{x_i} \cap B_r(p)) = \emptyset.$$ 

So $B_r(p) \subseteq K^c$, showing that $p$ is an interior point of $K^c$. We conclude that $K^c$ is open and thus $K$ is closed. \qed
2. Let \((s_n)\) be a sequence of non-zero real numbers, and assume that \((s_n)\) converges to a real number \(s \neq 0\). Prove that \((\frac{1}{s_n})\) converges to \(\frac{1}{s}\).

**Scratchwork:** To show that \(\frac{1}{s_n} \to \frac{1}{s}\), you must show that when \(n\) is large \(\left| \frac{1}{s_n} - \frac{1}{s} \right|\) is small. Since

\[
\left| \frac{1}{s_n} - \frac{1}{s} \right| = \frac{|s - s_n|}{|s_n s|}
\]

and we know we can make \(|s - s_n|\) small, the key is to make sure that \(|s_n s|\) is not too small.

**Solution:** Since \(s_n \to s\), there is \(M \in \mathbb{N}\) with \(\forall n \geq M \ |s_n - s| \leq \frac{1}{2}|s|\). For \(n \geq M\) the triangle inequality gives

\[
|s| \leq |s_n| + |s - s_n| < |s_n| + \frac{1}{2}|s|
\]

and thus \(|s_n| > \frac{1}{2}|s|\) (in the scratchwork this corresponds to making sure \(|s_n s|\) is not too small).

Let \(\epsilon > 0\). Since \(s_n \to s\) there is \(N \in \mathbb{N}\) with \(\forall n \geq N \ |s_n - s| < \frac{1}{2}|s|^2 \epsilon\). So for all \(n \geq \max(N, M)\) we have

\[
\left| \frac{1}{s_n} - \frac{1}{s} \right| = \left| \frac{s - s_n}{s_n s} \right| < \frac{|s - s_n|}{\frac{1}{2}|s|^2} < \epsilon.
\]

We conclude that \(\frac{1}{s_n} \to \frac{1}{s}\). \(\Box\)

**Note:** Another good tool for finding the values of limits is displayed in the solution to problem 3 on midterm 2.
3. Let \((a_n)\) be a bounded sequence of real numbers and set \(A = \{x \in \mathbb{R} : \exists N \forall n \geq N \ a_n \geq x\}\). Prove that \(\text{sup} \ A = \lim \inf a_n\).

**Solution:** Consider any convergent subsequence \((a_{n_k})\) and any \(x \in A\). Since \(x \in A\) there is \(N\) so that for all \(n \geq N\) we have \(a_n \geq x\). In particular, since \(n_k \geq k\) for all \(k\), we have \(a_{n_k} \geq x\) for all \(k \geq N\). Therefore \(\lim_{k \to \infty} a_{n_k} \geq x\). This shows that every subsequential limit of \((a_n)\) is an upperbound to \(A\). Since \(\lim \inf a_n\) is a subsequential limit of \((a_n)\), we find that \(\lim \inf a_n\) is an upperbound to \(A\).

Now we show that \(\lim \inf a_n\) is the least upperbound to \(A\). So consider any \(y < \lim \inf a_n\). We will show that \(y\) is not an upperbound to \(A\). Fix any \(x \in \mathbb{R}\) with \(y < x < \lim \inf a_n\). By a theorem we learned (Theorem 3.17 in the book), there is \(N\) so that for all \(n \geq N\) we have \(a_n \geq x\). This means \(x \in A\), and since \(x > y\) we find that \(y\) is not an upperbound to \(A\). We conclude that \(\lim \inf a_n\) is the least upperbound to \(A\), meaning \(\lim \inf a_n = \text{sup} \ A\). \(\square\)
4. Prove that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$.

**Solution:** Let $p > 1$. Since $p > 0$, for every $n \in \mathbb{Z}_+$ we have $\frac{1}{n^p} \geq 0$ and $\frac{1}{n^p} \geq \frac{1}{(n+1)^p}$.

From these two properties it follows (see Theorem 3.27) that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p}$ converges. Since $2^k \cdot \frac{1}{(2^k)^p} = (2^{1-p})^k$, we see that $\sum_{k=0}^{\infty} 2^k \cdot \frac{1}{(2^k)^p}$ is the geometric series $\sum_{k=0}^{\infty} (2^{1-p})^k$. Since $p > 1$, we have $2^{1-p} < 1$ and therefore the geometric series $\sum_{k=0}^{\infty} (2^{1-p})^k$ converges. We conclude that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges.
5. Let \( f : [0, \infty) \rightarrow \mathbb{R} \) be continuous and suppose that \( \lim_{x \to \infty} f(x) = 0 \). Prove that \( f \) is uniformly continuous.

**Comment:** We know that continuous functions are uniformly continuous on every compact set. So the restriction of \( f \) to any closed and bounded interval is uniformly continuous. The obstruction to \( f \) being uniformly continuous comes precisely from the fact that its domain is not bounded. However, the condition that \( \lim_{x \to \infty} f(x) = 0 \) will allow us to control the long-term behavior of \( f \). In our proof we will essentially break into two pieces – one piece will handle a neighborhood of \( +\infty \) while the other piece will handle a closed and bounded interval.

**Solution:** Let \( \epsilon > 0 \). Since \( \lim_{x \to \infty} f(x) = 0 \), there is \( M > 0 \) such that \( \forall x > M \ |f(x)| < \frac{\epsilon}{2} \) (this condition will handle uniform continuity for large values of \( x \)). Since \([0, M + 1]\) is compact, \( f \) is uniformly continuous on \([0, M + 1]\). So there is \( \delta_0 > 0 \) with

\[
\forall x_1, x_2 \in [0, M + 1] \ |x_1 - x_2| < \delta_0 \Rightarrow |f(x_1) - f(x_2)| < \epsilon.
\]

(We have now taken care of uniform continuity of \( f \) on the interval \([0, M + 1]\) and the interval \((M, \infty)\). The fact that these intervals overlap on a segment of length one is intentional and significant). Set \( \delta = \min(\delta_0, 1) \) (we use 1 here because 1 is the length of the overlap between \([0, M + 1]\) and \((M, \infty)\)). Now consider \( x_1, x_2 \in [0, \infty) \) with \( |x_1 - x_2| < \delta \). In particular, \( |x_1 - x_2| < 1 \) and thus either \( x_1, x_2 \in [0, M + 1] \) or \( x_1, x_2 \in (M, \infty) \) (the overlap and choice of \( \delta \) allows us to reduce to these two cases).

**Case 1:** \( x_1, x_2 \in [0, M + 1] \). Since \( |x_1 - x_2| < \delta \leq \delta_0 \), from the definition of \( \delta_0 \) we obtain

\[
|f(x_1) - f(x_2)| < \epsilon.
\]

**Case 2:** \( x_1, x_2 > M \). Then \( |f(x_1) - f(x_2)| \leq |f(x_1)| + |f(x_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \)

We conclude that \( f \) is uniformly continuous. \( \square \)