MATH 142A WINTER 2013 MIDTERM 1

Instructions: No books, notes, calculators, etc. are allowed during the exam. You may quote the theorems that we proved in class, or that are proved in the textbook, in your proofs, unless the problem says otherwise. Generally, do not quote the result of a homework exercise in your proof—if you need such a result you should go through the proof again.

1. (a). (2 pts) Carefully define what it means for a subset $S \subseteq \mathbb{R}$ to be dense.

(b). (3 pts) Carefully define what it means for a subset $S \subseteq \mathbb{R}$ to be closed.

(c). (5 pts) Give an example of a set $S \subseteq \mathbb{R}$ which is dense and bounded, or show that no such example exists.

(d). (5 pts) Show that if $S \subseteq \mathbb{R}$ is dense and closed, then $S = \mathbb{R}$.

2. (a). (5 pts) Prove that if a and b are *positive* real numbers, then $a^3 < b^3$ if and only if $a < b$.

(b). (10 pts) Show that $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$, using part (a) and the Archimedean property.

3. (10 pts) Let $S = \{x \in \mathbb{R} | x > 0 \text{ and } x^2 > x\}$. Prove that $\inf S$ exists and is equal to 1.

4. (10 pts) Let $\{a_n\}$ be a sequence of *integers*, in other words $a_n \in \mathbb{Z}$ for all $n \geq 1$. Suppose that $\{a_n\}$ converges. Prove that $\{a_n\}$ is eventually constant: that is, there is $m \in \mathbb{Z}$ and $N \in \mathbb{N}$ such that $a_n = m$ for all $n \geq N$.

Date: January 31, 2014.

Math 142A - Exam 1 Solutions

1.

(a) Carefully define what it means for a subset $S \subseteq \mathbb{R}$ to be dense.

Def. A subset $S \subseteq \mathbb{R}$ is dense if any interval (a, b) where $a < b$ contains a member of S. ■

(b) Carefully define what it means for a subset $S \subset \mathbb{R}$ to be closed.

Def. A subset $S \subseteq \mathbb{R}$ is closed if for every sequence $\{a_n\}$ in S, if a_n converges to a, then $a \in S$. ■

(c) Give an example of a set $S \subseteq \mathbb{R}$ which is dense and bounded, or show that no such example exists.

Proof. No such example exists. If S is bounded then there exists $M \in \mathbb{N}$ such that for any $s \in S$, $|s| \leq M$. In particular, the interval $(M, M+1)$ contains no element of S. Hence, S is not dense. ■

(d) Show that if $S \subseteq \mathbb{R}$ is dense and closed, then $S = \mathbb{R}$.

Proof. Let $a \in \mathbb{R}$, we show $a \in S$ so that $S = \mathbb{R}$. Since S is dense, a is the limit of a sequence in S (Proposition 2.19). Since S is closed, this implies $a \in S$. ■

2.

(a) Prove that if a and b are positive real numbers, then $a^3 < b^3$ if and only if $a < b$.

Proof. Suppose $a^3 < b^3$. Then $a^3 - b^3 = (a-b)(a^2 + ab + b^2) < 0$. But since a and b are positive, so is $a^2 + ab + b^2$. Thus $a - b < 0$ so we conclude $a < b$.

Conversely, if $a < b$ then $a - b < 0$. Again, $a^3 - b^3 = (a-b)(a^2 + ab + b^2) < 0$ so $a^3 < b^3$. ■

(b) Show that $\lim_{n\to\infty} \frac{1}{\sqrt[3]{n}} = 0$ using part (a) and the Archimedean property.

Proof. Let $\epsilon > 0$. By the Archimedean property, there exists an $N \in \mathbb{N}$ such that $1/N < \epsilon^3$. By part (a), $1/\sqrt[3]{N} < \epsilon$.

For all $n \geq N$, we have $1/n \leq 1/N$. Again by the above, for all $n \geq N$, we have $|1/\sqrt[3]{n}| = 1/\sqrt[3]{n} \leq 1/\sqrt[3]{N} < \epsilon$. So $1/\sqrt[3]{n}$ converges to 0. ■

3. Let $S = \{x \in \mathbb{R} \mid x > 0 \text{ and } x^2 > x\}$. Prove that $\inf S$ exists and is equal to 1.

Proof. Since $2 \in S$, $S \neq \emptyset$. Notice that S is bounded below by 1. To see this, suppose $y < 1$. Either $y < 0$ or $y \in (0, 1)$. If $y < 0$, then $y \notin S$. If $0 < y < 1$ then $y^2 < y$ so $y \notin S$. In either case, if $y < 1$ then $y \notin S$, so 1 is a lower bound for S.

By the completeness axiom, $\inf S$ exists. Suppose for contradiction that $\inf S > 1$. Then take any real number y in the interval $(1, \inf S)$ (for example, a rational is in the interval by density of $\mathbb{Q}$). Since $y > 1$, we have $y^2 > y$. So $y \in S$ but this contradicts $\inf S$ being a lower bound for S. Hence $\inf S \leq 1$. Further, since 1 is a lower bound for S, $\inf S \geq 1$. We conclude that $\inf S = 1$. ■

4. Let $\{a_n\}$ be a sequence of integers. Suppose that $\{a_n\}$ converges. Prove that $\{a_n\}$ is eventually constant.

Proof. Suppose a_n converges to a. Then there exists some $N \in \mathbb{N}$ such that for all $n \geq N$, $|a_n - a| < 1/2$. That is, $a_n \in (a - 1/2, a + 1/2)$. But by Theorem 1.8, there is at most one integer in the interval $(a - 1/2, a + 1/2)$. Since for all $n \geq N$, $a_n \in \mathbb{Z}$, there must be a unique integer $m \in (a - 1/2, a + 1/2)$. So for all $n \geq N$, $a_n = m$.

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