Name: ______________________________ **PID:** ________________

Math 142A
Midterm Exam 1
February 1, 2008

Turn off and put away your cell phone.
No calculators or any other electronic devices are allowed during this exam.
You may use one page of notes, but no books or other assistance on this exam.
Read each question carefully, answer each question completely, and show all of your work.
Write your solutions clearly and legibly; no credit will be given for illegible solutions.
If any question is not clear, ask for clarification.

#	Points	Score
1	6	
2	6	
3	6	
4	6	
Σ	24	

1. Let $S = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}$.

    (a) Show that $\sqrt{2}$ is the infimum of S.

    (b) Show that S has no minimum.

2. Recall the following:

Definition: A sequence $\{a_n\}$ *converges* if and only if there is a number a such that for every $\epsilon > 0$ there is a natural number N such that for every index $n \geq N$, $|a_n - a| < \epsilon$.

(a) Write a clear statement defining what it means that a sequence $\{a_n\}$ does *not* converge.

(b) Using the negation of the definition of convergence, prove directly that the sequence $\{a_n\}$ defined by $a_n = (-1)^n$ does not converge.

3. Prove that the interval $(0, 1]$ is not closed.

4. Consider the sequence $\{a_n\}$ defined as follows:

$$a_n = \begin{cases} 2 & \text{if } n = 1, \\ 1 + \dfrac{1}{a_{n-1}} & \text{if } n > 1. \end{cases}$$

(a) Prove that $\{a_n\}$ is monotonic.

(b) Prove that $\{a_n\}$ is bounded.

(c) Does $\{a_n\}$ converge? Justify your answer.

Math 142A
Midterm Exam 1 Solution

1. Let $S = \{x \in \mathbb{Q} \mid x > \sqrt{2}\}$.

   (a) Show that $\sqrt{2}$ is the infimum of S.

   Since $x > \sqrt{2}$ for every $x \in S$, $\sqrt{2}$ is a lower bound for S. Suppose $y > \sqrt{2}$. Then, there is a rational number r such that $\sqrt{2} < r < y$ since the rational numbers are dense. Hence, $r \in S$ with $r < y$. It follows that every number $y > \sqrt{2}$ is not a lower bound for S. Therefore, $\sqrt{2}$ is the infimum of S.

   (b) Show that S has no minimum.

   If $x \in S$, then $x > \sqrt{2}$ and thus not a lower bound for S. Therefore, S has no minimum.

2. Recall the following:

   Definition: A sequence $\{a_n\}$ *converges* if and only if there is a number a such that for every $\epsilon > 0$ there is a natural number N such that for every index $n \geq N$, $|a_n - a| < \epsilon$.

   (a) Write a clear statement defining what it means that a sequence $\{a_n\}$ does *not* converge.

   A sequence $\{a_n\}$ does not converge if and only if for every number a there exists $\epsilon > 0$ such that for every natural number N there exists an index $n \geq N$ such that $|a_n - a| \geq \epsilon$.

   (b) Using the negation of the definition of convergence, prove directly that the sequence $\{a_n\}$ defined by $a_n = (-1)^n$ does not converge.

   Let $a \in \mathbb{R}$. Then $|(-1)^n - a| + |(-1)^{n+1} - a| \geq |(-1)^n - a + a - (-1)^{n+1}| = 2$, for every index n. Thus, for every index n, either $|(-1)^n - a| \geq 1$ or $|(-1)^{n+1} - a| \geq 1$. It follows that $\{a_n\}$ does not converge.

3. Prove that the interval $(0, 1]$ is not closed.

   $\{\frac{1}{n}\}$ is a sequence in $(0, 1]$ converging to 0 and $0 \notin (0, 1]$. It follows that $(0, 1]$ is not closed.

4. Consider the sequence $\{a_n\}$ defined as follows:

$$a_n = \begin{cases} 2 & \text{if } n = 1, \\ 1 + \dfrac{1}{a_{n-1}} & \text{if } n > 1. \end{cases}$$

(a) Prove that $\{a_n\}$ is monotonic.

$a_1 = 2$, $a_2 = \frac{3}{2}$ and $a_3 = \frac{5}{3}$. Since $a_1 > a_2$ and $a_2 < a_3$, the statement that $\{a_n\}$ is monotone is pure balderdash.

(b) Prove that $\{a_n\}$ is bounded.

Observe that $a_{n+2} = 1 + \frac{1}{a_{n+1}} = 1 + \frac{1}{1+\frac{1}{a_n}} = \frac{2a_n+1}{a_n+1}$. Thus, if we define the sequences $\{b_n\}$ and $\{c_n\}$ by $b_n = a_{2n-1}$ and $c_n = a_{2n}$, we have

$$b_1 = 2; \qquad b_{n+1} = \frac{2b_n + 1}{b_n + 1}$$
$$c_1 = \frac{3}{2}; \qquad c_{n+1} = \frac{2c_n + 1}{c_n + 1}$$

Then, $b_n^2 - b_n - 1 > 0$ and $c_n^2 - c_n - 1 < 0$ for every index n since this is true for $n = 1$ and if $b_k^2 - b_k - 1 > 0$ and $c_k^2 - c_k - 1 < 0$ for some index k, then

$$b_{k+1}^2 - b_{k+1} - 1 = \left(\frac{2b_k + 1}{b_k + 1}\right)^2 - \left(\frac{2b_k + 1}{b_k + 1}\right) - 1 = \frac{b_k^2 - b_k - 1}{(b_k + 1)^2} > 0$$
$$c_{k+1}^2 - c_{k+1} - 1 = \left(\frac{2c_k + 1}{c_k + 1}\right)^2 - \left(\frac{2c_k + 1}{c_k + 1}\right) - 1 = \frac{c_k^2 - c_k - 1}{(c_k + 1)^2} < 0.$$

Thus,

$$b_{n+1} - b_n = \left(\frac{2b_n + 1}{b_n + 1}\right) - b_n = -\frac{b_n^2 - b_n - 1}{b_n + 1} < 0$$
$$c_{n+1} - c_n = \left(\frac{2c_n + 1}{c_n + 1}\right) - c_n = -\frac{c_n^2 - c_n - 1}{c_n + 1} > 0,$$

from which it follows that $\{b_n\}$ is monotonically decreasing and $\{c_n\}$ is monotonically increasing. Since the terms of $\{b_n\}$ and $\{c_n\}$ are clearly positive for all indices n, we can conclude that $\{b_n\}$ is bounded below by α and $\{c_n\}$ is bounded above by α, where $\alpha = \frac{1+\sqrt{5}}{2}$ is the positive solution to $x^2 - x - 1 = 0$. Thus, both $\{b_n\}$ and $\{c_n\}$ converge by the monotone convergence theorem. Let β be the limit of $\{b_n\}$ and let γ be the limit $\{c_n\}$. Then, $\beta = 1 + \frac{1}{\beta}$ and $\gamma = 1 + \frac{1}{\gamma}$. Hence, $\beta^2 - \beta - 1 = 0$ and $\gamma^2 - \gamma - 1 = 1$, from which it follows that $\beta = \gamma = \alpha$.

(c) Does $\{a_n\}$ converge? Justify your answer.

Yes, $\{a_n\}$ converges to the common limit of $\{b_n\}$ and $\{c_n\}$, namely $\alpha = \frac{1+\sqrt{5}}{2}$.