MATH 142A WINTER 2013 SAMPLE EXAM 2

1. What the exam covers

The exam 2 syllabus includes 2.3 on the Monotone Convergence Theorem and covers up through Section 3.5. I decided that Sections 3.6-3.7 will not be covered on the exam. The subject is cumulative, so you are still responsible for knowing the material we covered before Midterm 1, but the exam will be designed to test your knowledge of Sections 2.3-3.5 specifically.

2. Some definitions/theorems that will be provided on the exam for your reference

We had several competing definitions of continuity and uniform continuity, and I understand sorting these out can be confusing. Sequential compactness is also a tricky concept, and it seems similar to closedness at first glance. I want you to concentrate on understanding what the definitions and theorems mean and how to use them, rather than spending time on memorization, so the following definitions (and only these) will be provided on the exam paper for your reference.

A set S of real numbers is *closed* if every convergent sequence $\{a_n\}$ of elements of S converges to an element of S. A set S of real numbers is *sequentially compact* if every sequence $\{a_n\}$ of elements of S has a convergent subsequence which converges to an element of S.

A function $f : D \to \mathbb{R}$ is *continuous* at a point $x_0 \in D$ if for all sequences $\{x_n\}$ of elements in D such that $\lim_{n\to\infty} x_n = x_0$, one has $\lim_{n\to\infty} f(x_n) = f(x_0)$. One says that f *satisfies the* $\epsilon - \delta$ *criterion* at $x_0 \in D$ if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x \in D$ with $|x - x_0| < \delta$, one has $|f(x) - f(x_0)| < \epsilon$. Theorem: A function f is continuous at x_0 if and only if f satisfies the $\epsilon - \delta$ criterion at x_0. A function $f : D \to \mathbb{R}$ is *continuous* if it is continuous at x_0 for all points x_0 in D.

A function $f : D \to \mathbb{R}$ is *uniformly continuous* if for all sequences $\{u_n\}$, $\{v_n\}$ of elements of D such that $\lim_{n\to\infty}(u_n - v_n) = 0$, one has $\lim_{n\to\infty}(f(u_n) - f(v_n)) = 0$. The function f satisfies the $\epsilon - \delta$ *criterion on the domain* D if for all $\epsilon > 0$ there exists a $\delta > 0$ such that for all $u, v \in D$ with $|u - v| < \delta$, one has $|f(u) - f(v)| < \epsilon$. Theorem: f is uniformly continuous if and only if f satisfies the $\epsilon - \delta$ criterion on the domain D.

3. Sample Exam

1. Suppose a set S of real numbers is sequentially compact. Prove that S is bounded.

2. Suppose that $\{a_n\}$ and $\{b_n\}$ are sequences of *positive* real numbers with $a_n \leq b_n$ for all $n \in \mathbb{N}$. Let $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} b_k$ and consider the sequences $\{s_n\}$ and $\{t_n\}$. Show that if $\{t_n\}$ converges, then $\{s_n\}$ converges.

3. Suppose that the functions $g : [a, b] \to \mathbb{R}$ and $h : [a, b] \to \mathbb{R}$ are continuous. Show that if $h(a) < g(a)$ and $h(b) > g(b)$, then there is $x \in (a, b)$ such that $g(x) = h(x)$.

4. Prove that if D is a bounded set of real numbers, then $f : D \to \mathbb{R}$ defined by $f(x) = x^2$ is uniformly continuous.

4. Solutions to the sample exam

1. Suppose that S is sequentially compact but unbounded. Then either S has no upper bound or no lower bound. Suppose that S has no upper bound. Then for each $n \in \mathbb{N}$, we can choose $a_n \in S$ such that $a_n \geq n$. Since S is sequentially compact, $\{a_n\}$ has a subsequence $\{b_k\}$ where $b_k = a_{n_k}$ for some $1 \leq n_1 < n_2 < \dots$, such that $\{b_k\}$ converges to a point of S. But $b_k = a_{n_k} \geq n_k \geq k$ for each k, so the sequence $\{b_k\}$ also fails to have an upper bound. On the other hand, we proved that every convergent sequence is bounded. Thus $\{b_k\}$ does not even converge, and this is a contradiction. If S has no lower bound a similar proof applies.

2. Since the a_n and b_n are all positive numbers, $s_{n+1} = s_n + a_{n+1} > s_n$ and $t_{n+1} = t_n + b_{n+1} > t_n$ for all $n \in \mathbb{N}$. Thus $\{s_n\}$ and $\{t_n\}$ are both monotonically increasing sequences. If $\{t_n\}$ converges, then it is bounded since we proved that any convergent sequence is bounded. Say c is an upper bound for the sequence $\{t_n\}$. Since $a_n \leq b_n$ for all $n \in \mathbb{N}$, clearly $s_n \leq t_n$ for all $n \in \mathbb{N}$. Thus the sequence $\{s_n\}$ also has c as an upper bound. Thus $\{s_n\}$ has an upper bound. Since $\{s_n\}$ is monotonically increasing, it obviously has a lower bound (for instance, s_1). Since $\{s_n\}$ is bounded and monotonically increasing, it converges by the Monotone Convergence Theorem.

3. Consider the function $h - g$ where $[h-g](x) = h(x) - g(x)$. The hypothesis $h(a) < g(a)$ implies that $[h-g](a) < 0$ and the hypothesis $h(b) \geq g(b)$ implies that $[h-g](b) > 0$. Then by the Intermediate Value Theorem, there exists x with $a < x < b$ such that $[h-g](x) = 0$. Then $h(x) = g(x)$, as required.

4. For any u and v in D, we have $|u^2 - v^2| = |u+v||u-v|$ by the difference of powers formula. Since D is bounded, there is $c > 0$ such that $|x| \leq c$ for all $x \in D$. Then since $u, v \in D$, we have that $|u+v| \leq |u| + |v| \leq 2c$. Thus $|u^2 - v^2| \leq 2c|u-v|$ for all $u, v \in D$.

Now given any $\epsilon > 0$, choose $\delta = \epsilon/(2c)$. Then for any $u, v \in D$ such that $|u - v| < \delta$, we have $|f(u) - f(v)| = |u^2 - v^2| \leq 2c|u-v| < 2c\delta = \epsilon$. This shows that f satisfies the $\epsilon - \delta$ condition on the domain D, which by a theorem is the same as uniform continuity of f.