**Name:** ______________________________ **PID:** ________________

**Math 142A**
**Final Examination**
**March 19, 2008**

*Turn off and put away your cell phone.*
*No calculators or any other electronic devices are allowed during this exam.*
*You may use one page of notes, but no books or other assistance on this exam.*
*Read each question carefully, answer each question completely, and show all of your work.*
*Write your solutions clearly and legibly; no credit will be given for illegible solutions.*
*If any question is not clear, ask for clarification.*

| # | Points | Score |
|---|---|---|
| **1** | 6 | |
| **2** | 6 | |
| **3** | 6 | |
| **4** | 6 | |
| **5** | 6 | |
| **6** | 6 | |
| **Σ** | 36 | |

1. Define the sequence $\{x_n\}$ by

$$\begin{cases} x_1 = 1 \\ x_{n+1} = \sqrt{x_n + 5} \text{ for } n \geq 1 \end{cases}$$

(a) Show that $\{x_n\}$ increases monotonically.

(b) Show that $\{x_n\}$ converges.

2. Let $a$ be a real number and let $S = \{x \in \mathbb{Q} \mid x < a\}$, where $\mathbb{Q}$ is the set of rational numbers. Prove that $a = \sup S$.

3. Let $\{a_n\}$ be a monotonically increasing sequence that has a bounded subsequence $\{a_{n_k}\}$. Prove that $\{a_n\}$ converges.

4. Let $I$ be a neighborhood of $x_0$ and suppose that the function $g : I \to \mathbb{R}$ is differentiable. Define

$$h(x) = \begin{cases} \frac{g(x)-g(x_0)}{x-x_0} & \text{if } x \neq x_0, \\ g'(x_0) & \text{if } x = x_0. \end{cases}$$

Prove that $h : I \to \mathbb{R}$ is continuous.

5. Let $I$ be a neighborhood of $x_0$ and let $f : I \to \mathbb{R}$ be differentiable. Prove that if $f(x) \leq f(x_0)$ for all $x \in I$, then $f'(x_0) = 0$.

6. A function $f : D \to \mathbb{R}$ is **C-Lipschitz** provided $|f(x) - f(y)| \leq C|x - y|$ for all $x, y \in D$. Prove that if a $C$-Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then $|f'(x)| \leq C$ for all $x \in \mathbb{R}$.