MATH 142A WINTER 2013 SAMPLE FINAL EXAM

1. Exam format

The final exam is cumulative, but will emphasize the topics we have covered since the second midterm.

Note: I have decided to allow each student to bring one page (a standard 8.5 by 11 sheet) of notes, both sides, to the exam. On the second midterm I provided some definitions for you, and this is meant to serve the same purpose except you get to choose what goes on it. The purpose of this is for you write down some of the definitions and theorems you find especially difficult, so that in your studying you can concentrate on understanding what these definitions and theorems are about, rather than rote memorization. Cramming this page full of every possible thing you can think of is counterproductive, as if you are not well-prepared you will spend too much time trying to find things on your notes. If you are confident in your understanding of the material of the course, you may need to write very little on this sheet of notes. No books, notes (other than the one allowed page), calculators, etc. are allowed at the final.

Because the final needs to cover an entire quarter's worth of material, only some selection of the topics we studied can be tested by any single exam. So you should treat the sample exam below as just that, a sample. There will be topics that appear on the sample exam and not at all on the final exam, and vice versa.

You should bring a blue book to the final, and your UCSD ID card. In a class this large, unfortunately there are students that none of us have gotten to know personally. So we will check student ID's at the final to ensure that those taking the exam are the students who are registered for the class.

2. SAMPLE EXAM

1. Recall that a set S is *dense* if for all $a < b$, there exists $x \in S$ with $x \in (a, b)$. Call the set S *sequentially dense* if given any real number r, there exists a sequence $\{a_n\}$ of elements of S such that $\lim_{n\to\infty} a_n = r$.

Prove that is S a dense set, then S is sequentially dense. (Note: this is part of a theorem in the text. You cannot quote that theorem in this problem, rather I want you to prove this statement directly.)

2. Show that if a subset S of $\mathbb{R}$ is closed and bounded, then S is sequentially compact.

3. Suppose that $\{a_n\}$ is a monotonically increasing sequence of real numbers. Suppose that $\{a_n\}$ has a convergent subsequence. Then show that $\{a_n\}$ is also convergent.

4. Suppose that $f : [0, 1] \to \mathbb{R}$ is continuous and that its image consists entirely of rational numbers. Prove that f is a constant function.

5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are uniformly continuous and bounded functions. Prove that $fg : \mathbb{R} \to \mathbb{R}$ is also uniformly continuous. (Hint: $(fg)(v) - (fg)(u) = f(v)[g(v) - g(u)] + g(u)[f(v) - f(u)]$.)

6. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = \sqrt{x}$. Prove *directly from the definition of the derivative* that $f'(x)$ exists for all $x \in (0, \infty)$, with $f'(x) = \dfrac{1}{2(\sqrt{x})}$.

7. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable and that there is a positive number c such that $f'(x) \geq c$ for all $x \in \mathbb{R}$. Prove that $f(x) \geq f(0) + cx$ for all $x \geq 0$.

3. Solutions to the sample exam

1. Let S be dense. Given $r \in \mathbb{R}$, for each $n \geq 1$ consider the interval $(r - 1/n, r)$. By the definition of density, for each $n \geq 1$ we can choose an element $x_n \in S$ with $x_n \in (r - 1/n, r)$.

Thus $\{x_n\}$ is a sequence of elements of S. We claim that $\lim_{n\to\infty} x_n = r$. Given $\epsilon > 0$, choose $N \in \mathbb{N}$ such that $N > 1/\epsilon$ (by the Archimedean principle). Then for all $n \geq N$, we have $n \geq 1/\epsilon$ and so $1/n \leq \epsilon$. since $r - 1/n < x_n < r$, we have $|r - x_n| = r - x_n < 1/n \leq \epsilon$, for all $n \geq N$. This shows by definition that $\lim_{n\to\infty} x_n = r$ as claimed.

Since r was arbitrary, it follows that every real number is the limit of a sequence of elements of S, so that S is sequentially dense, as desired.

2. Assume that S is closed and bounded. Let $\{a_n\}$ be any sequence of elements in S. Since S is bounded, the sequence $\{a_n\}$ is bounded, and so by a theorem we proved, $\{a_n\}$ has a convergent subsequence $\{b_k\}$, where $b_k = a_{n_k}$ for some $1 \leq n_1 < n_2 < n_3 < \dots$. Since $\{b_k\}$ is a convergent sequence consisting of elements of S, its limit $\lim_{k\to\infty} b_k$ must belong to S, by the definition of closed set.

We have shown that any sequence of elements of S has a convergent subsequence whose limit is also an element of S. By definition, S is sequentially compact.

3. By hypothesis, $\{a_n\}$ has a convergent subsequence $\{b_k\}$, where $b_k = a_{n_k}$ for some $1 \leq n_1 < n_2 < \dots$. Since $\{b_k\}$ is a convergent sequence, it is bounded by a theorem we proved. Thus there is some constant C such that $b_k \leq C$ for all $k \geq 0$. Now given any $n \geq 0$, we have $n_k \leq n < n_{k+1}$ for some $k \geq 0$. Then since $\{a_n\}$ is monotonically increasing, $a_{n_k} \leq a_n \leq a_{n_{k+1}}$. This shows that $a_n \leq a_{n_{k+1}} \leq C$. So $\{a_n\}$ also has C as an upper bound. Since $\{a_n\}$ is monotonically increasing, it also has a lower bound of a_1. So $\{a_n\}$ is bounded and monotonically increasing. By the monotone convergence theorem, $\{a_n\}$ is convergent.

4. Let f be continuous and have image contained in $\mathbb{Q}$. Suppose that f is not constant. Then it takes on at least two different values, so we can choose points x_1 and x_2 of $[0, 1]$ with $x_1 < x_2$ such that $f(x_1) \neq f(x_2)$. By hypothesis, $f(x_1)$ and $f(x_2)$ are rational. We proved that the irrational numbers are dense in $\mathbb{R}$, so every open interval contains an irrational number. So either $f(x_1) < r < f(x_2)$ for some irrational r, or $f(x_2) < r < f(x_1)$ for some

irrational r (depending on which of $f(x_1)$ and $f(x_2)$ is larger). By the intermediate value theorem, there is some x_0 with $x_1 < x_0 < x_2$ such that $f(x_0) = r$. This contradicts the hypothesis that the image of f is contained in the rational numbers. Hence f is constant.

5. We use the $\epsilon - \delta$ formulation of uniform continuity. (The proof using the sequential definition is similar.) Fix some $\epsilon > 0$. Since f and g are bounded, there are constants $C_1 > 0$ and $C_2 > 0$ such that $|f(x)| \leq C_1$ and $|g(x)| \leq C_2$ for all $x \in \mathbb{R}$. By putting $C = \max(C_1, C_2)$ we can assume that $|f(x)| \leq C$ and $|g(x)| \leq C$ for all $x \in \mathbb{R}$.

Since f is uniformly continuous, there exists $\delta_1 > 0$ such that $|f(v) - f(u)| \leq \epsilon/(2C)$ for all u, v with $|v - u| \leq \delta_1$. Similarly, since g is uniformly continuous, there exists $\delta_2 > 0$ such that $|g(v) - g(u)| \leq \epsilon/(2C)$ for all u, v with $|v - u| \leq \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$. Then for all u, v with $|v - u| \leq \delta$, using the formula in the hint and the triangle inequality we have

$$|fg(v) - fg(u)| \leq |f(v)||g(v) - g(u)| + |g(u)||f(u) - f(v)| \leq C\epsilon/(2C) + C\epsilon/(2C) = \epsilon.$$

By the $\epsilon - \delta$ formulation of uniform continuity, this shows that fg is uniformly continuous.

6. We use the formula $(\sqrt{u} - \sqrt{v})(\sqrt{u} + \sqrt{v}) = u - v$, which is a special case of the formula for the difference of two squares.

Let $x_0 \in (0, \infty)$. By definition, $f'(x_0)$, if it exists, is equal to

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0}.$$

By the formula above, the functions of x given by $\dfrac{\sqrt{x} - \sqrt{x_0}}{x - x_0}$ and $\dfrac{1}{\sqrt{x} + \sqrt{x_0}}$ are the same at all values of x other than x_0 (where the first function is undefined). The limit of a function at x_0 is the same regardless of the value of the function (if any) at x_0, so

$$\lim_{x \to x_0} \frac{\sqrt{x} - \sqrt{x_0}}{x - x_0} = \lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}}.$$

But the function $\dfrac{1}{\sqrt{x} + \sqrt{x_0}}$ is a continuous function at all points x in the domain, including at x_0 (we proved that $\sqrt{x}$ is continuous, sums of continuous functions are continuous, and ratios of continuous functions are continuous as long as the denominator is nonzero. Note that since $x_0 > 0$, $\sqrt{x} + \sqrt{x_0}$ is never 0 for $x > 0$.)

The limit of a function which is continuous at x_0 is given by evaluating the function at x_0, so that

$$\lim_{x \to x_0} \frac{1}{\sqrt{x} + \sqrt{x_0}} = \frac{1}{\sqrt{x_0} + \sqrt{x_0}} = \frac{1}{2\sqrt{x_0}}.$$

7. Apply the mean value theorem to the restriction of f to the closed bounded interval $[0, x]$, for any $x > 0$. It shows that there exists x_0 with $x_0 \in (0, x)$ such that $f'(x_0) = \dfrac{f(x) - f(0)}{x - 0} = f'(x_0)$. Since $f'(x_0) \geq c$, we get $\dfrac{f(x) - f(0)}{x} = f'(x_0) \geq c$, which implies that $f(x) - f(0) \geq cx$ since $x > 0$. Thus $f(x) \geq f(0) + cx$. Since $x > 0$ was arbitrary, the result is proved.