1. Let \( K = \mathbb{F}_p(x, y) \) be rational functions in two variables over the field \( \mathbb{F}_p \) of \( p \) elements. Let \( F = \mathbb{F}_p(x^p, y^p) \).
   (a). Show that \( K/F \) is purely inseparable, and that \([K : F] = p^2\).
   (b). Show that there are infinitely many intermediate subfields between \( F \) and \( K \), and conclude that the extension \( K/F \) does not have a primitive element.

2. In this problem you show that \( G = \text{Aut(\mathbb{R})} \) is trivial.
   (a). Show that every element of \( G \) fixes \( \mathbb{Q} \) pointwise; that is, \( \text{Aut(\mathbb{R})} = \text{Aut(\mathbb{R}/\mathbb{Q})} \).
   (b). Let \( \sigma \in G \). Prove that \( \sigma \) takes squares to squares and hence takes the set of positive numbers to itself. Using this conclude that \( a < b \) implies \( \sigma(a) < \sigma(b) \).
   (c). Prove that \( \sigma \) is a continuous function.
   (d). Conclude that \( \sigma \) is the identity.

3. Let \( F \subseteq K \) be a Galois extension and let \( F \subseteq E \subseteq K \) and \( F \subseteq L \subseteq K \) be intermediate fields. Show that there is an isomorphism \( \theta : E \to L \) which restricts to the identity on \( F \) if and only if the subgroups \( \text{Gal}(K/L) \) and \( \text{Gal}(K/E) \) of \( G = \text{Gal}(K/F) \) are conjugate subgroups in \( G \).

4. Let \( F \subseteq K \) be an algebraic extension, not necessarily of finite degree.
   (a). Let \( E \) be the set of elements \( \alpha \in K \) such that \( \alpha \) is separable over \( F \). Show that \( E \) is a subfield of \( K \) containing \( F \). (Hint: Show for any \( \alpha, \beta \in E \) that \( F \subseteq F(\alpha, \beta) \) is a separable extension. To do this, construct a Galois extension of \( F \) containing \( \alpha \) and \( \beta \).)
   (b). With the same notation as in (a), show that \( K/E \) is purely inseparable. Thus any algebraic extension decomposes into a separable extension followed by a purely inseparable extension.
   (c). Suppose that \( F \subseteq L \subseteq K \) where \( K/F \) is an algebraic extension. Show that \( K/F \) is separable if and only if \( L/F \) and \( K/L \) are separable.
5. Let $p_1, \ldots, p_n$ be different prime numbers and let $E = \mathbb{Q}(\sqrt{p_1}, \ldots, \sqrt{p_n})$ as a subfield of $\mathbb{R}$.

(a). Show that $E/\mathbb{Q}$ is Galois and that $\text{Gal}(E/\mathbb{Q})$ is elementary Abelian of order $2^n$. (Hint: Show that the fields $\mathbb{Q}(\sqrt{k})$ are all different as $k$ runs over the $2^n - 1$ different products of distinct members of the set \{p_1, \ldots, p_n\}.)

(b). Show that $E = \mathbb{Q}(\alpha)$, where $\alpha = \sqrt{p_1} + \sqrt{p_2} + \cdots + \sqrt{p_n}$. (Hint: determine how the $2^n$ elements of the Galois group $G$ act on the elements $\sqrt{p_1}, \ldots, \sqrt{p_n}$. Then show that the orbit of $\alpha$ under $G$ contains $2^n$ different elements.)

6. Let $p$ be a prime. Let $K$ be the splitting field of $f(x) = x^p - 2$ over $\mathbb{Q}$. Let $\zeta$ be any primitive $p$th root of 1 in $\mathbb{C}$, and let $\alpha = \sqrt[8]{2} \in \mathbb{C}$.

Show that $K = \mathbb{Q}(\zeta, \alpha)$ and that $[K : F] = p(p - 1)$. Let $G = \text{Gal}(K/\mathbb{Q})$ be the Galois group. Show that $G$ is isomorphic to the semidirect product $\mathbb{Z}_p \rtimes (\mathbb{Z}_p)^*$, where $(\mathbb{Z}_p)^*$ is the group of units of $\mathbb{Z}_p$, and $\phi : (\mathbb{Z}_p)^* \rightarrow \text{Aut}(\mathbb{Z}_p)$ is the natural identification.