We can keep doing this until $r = 0$.

Now we have $P(x,y,z) = \lambda z^m + y R(x,y,z)$

since $P(0,0,1) = 0$, we have $\lambda = 0$

So $P(x,y,z) = y R(x,y,z)$

$Q(x,y,z) = Q(x,0,z) + y S(x,y,z)$

Now $I_p(P,Q) = I_p(y,Q) + I_p(R,Q)$ by (v)

$\| $ by (vi)

$I_p(y, Q(x,0,z))$

$Q(x,0,z)$ is homogeneous in 2 variables, so it is a product of linear factors in x and z (they are independent of y)

So $I_p(y, Q(x,0,z)) = \deg Q$ by (v) and (iv)

So $k = I_p(P,Q) = \deg Q + I_p(R,Q)$

$\deg Q > 0$ so $I_p(R,Q) < k$

By the induction hypothesis, $I_p(R, Q)$ can be computed from (i), ..., (vi). So $I_p(P, Q)$ can also be computed. $\square$

Proof of Bézout's theorem:

Choose coordinates so that Ⓐ Ⓑ Ⓒ hold.

Then by Lemmas 3.4, 3.7, $\mathcal{R}_{P,Q} = \prod_{i=1}^{k} (c_i y - b_i z)^{e_i}$

where (b_i, c_i) and (b_j, c_j) are not proportional for $i \neq j$.

Recall: $\forall i \quad \mathcal{R}_{P(x, b_i, c_i), Q(x, b_i, c_i)} = 0 = \mathcal{R}_{P,Q}(b_i, c_i)$

$\iff \exists!\, a_i$ s.t. $P(a_i, b_i, c_i) = Q(a_i, b_i, c_i) = 0$

Note: we have uniqueness of a_i because otherwise, we would have (a_i, b_i, c_i) & $(a'_i, b_i, c_i) \in C \cap D$ are collinear with $(1, 0, 0) = (a_i - a'_i, 0, 0)$ if $a_i \neq a'_i$

We saw in the proof of Theorem 3.18 that

$$I_{(a_i, b_i, c_i)}(C, D) = e_i$$

By Lemma 3.7. $\deg \mathcal{R}_{P,Q} = mn$

Also $\deg \mathcal{R}_{P,Q} = e_1 + \cdots + e_k = \sum_{i=1}^{k} e_i$

So $\sum_{p \in C \cap D} I_p(P, Q) = \sum_{i=1}^{k} e_i = \deg \mathcal{R}_{P,Q} = mn$ $\square$

<u>Lemma 3.24</u>: If $p \in C \cap D$ is a singular point of C, then $I_p(C, D) > 1$.