As we saw before, if $I_p(P, x) \geq 3$, then

$y^3 \mid P(0, y, z)$ and $\mathcal{H}_P(p) = 0$.

<u>The converse:</u> ?

Assume $\mathcal{H}_P(p) = 0$, what can we say?

As before, assume $p = (0,0,1)$ and $L = \{x = 0\}$ is tangent to C at p, so that $\frac{\partial P}{\partial y}(0,0,1) = \frac{\partial P}{\partial z}(0,0,1) = 0$

and we have $\mathcal{R}_{P,x} = P(0,y,z)$ & $y^2 \mid P(0,y,z)$.

<u>Lemma 3.30</u>: Suppose $m = \deg P > 1$. Then

$$z^2 \mathcal{H}_P(x,y,z) = (m-1)^2 \begin{vmatrix} P_{xx} & P_{xy} & P_x \\ P_{yx} & P_{yy} & P_y \\ P_x & P_y & \frac{mP}{m-1} \end{vmatrix}$$

where $P_{xy} = \frac{\partial^2 P}{\partial x \partial y}$ etc.

<u>Proof</u>: Recall Euler's relation:

$$m P = x P_x + y P_y + z P_z$$

$$(m-1) P_x = x P_{xx} + y P_{xy} + z P_{xz}$$

$$(m-1) P_y = x P_{xy} + y P_{yy} + z P_{yz}$$

$$(m-1) P_z = x P_{xz} + y P_{yz} + z P_{zz}$$

multiply the last column by z and add x times the first column + y times the second column, then do the same with the rows to obtain the result. $\square$

Apply the lemma: At $p = (0,0,1)$ we have

$$0 = \mathcal{H}_P(p) = (m-1)^2 \begin{vmatrix} P_{xx}(p) & P_{xy}(p) & P_x(p) \\ P_{xy}(p) & P_{yy}(p) & 0 \\ P_x(p) & 0 & 0 \end{vmatrix} = -(m-1)^2 P_x^2 P_{yy}$$

So $\begin{cases} \text{either } P_x(p)=0 \text{ which means } p \text{ is a singular point of } C \\ \text{or } P_{yy}(p)=0 \text{ which means } y^3 \mid \underset{\underset{Res_{P,x}(y,z)}{\parallel}}{P(0,y,z)} \end{cases}$

So either p is a singular point of C

or $I_p(C,L) \geq 3$.

So we have the conclusion: $\exists$ line L s.t.

$I_p(C,L) \geq 3 \iff \mathcal{H}_P(p)=0$.

i.e., p is a point of inflection $\iff \mathcal{H}_P(p)=0$

<u>Remark:</u> Euler's relations for P_x, P_y, P_z show that if p is singular on C, we have $\mathcal{H}_P(p)=0$

So we can rephrase the conclusion:

$$C \cap Z(\mathcal{H}_p) = \{\text{nonsingular points of inflection}\} \cup \{\text{singular points}\}$$

<u>First application:</u>

<u>Corollary 3.34:</u> Suppose C is a nonsingular cubic. Then $\exists$ coordinates in which the equation of C is

$$y^2 z = x(x-z)(x-\lambda z) \quad \text{for some } \lambda \in \mathbb{C} \setminus \{0,1\}.$$

<u>Proof:</u> By the above $C \cap Z(\mathcal{H}_p) = \{\text{points of infection}\}$

By Bézout $C \cap Z(\mathcal{H}_p) \neq \phi$ because $\deg C = 3$

$$\deg \mathcal{H}_p = 3(m-2) = 3$$

So C has at least one point of inflection.

Suppose $p = (0,1,0)$ is a point of inflection of C and $\{z=0\}$ is the tangent line to C at p.

Write $P(x,y,z) = P(x,0,z) + y\,Q(x,y,z)$

restrict to the line $z=0$: $P(x,y,0) = P(x,0,0) + y\,Q(x,y,0)$

$= x^3$ up to a scalar.

because $C \cap L = \{p\}$ with multiplicity 3

$\Rightarrow Q(x,y,0) = 0 \Rightarrow z \mid Q(x,y,z)$

$\Rightarrow \exists\ \alpha, \beta, \gamma \in \mathbb{C}$ s.t. $Q(x,y,z) = z(\alpha x + \beta y + \gamma z)$.

Recap: $P(x,y,z) = P(x,0,z) + yz(\alpha x + \beta y + \gamma z)$

$0 \neq \frac{\partial P}{\partial z}(0,1,0) = \beta$

replace y with $y_1 := y + \frac{\alpha x + \gamma z}{2\beta}$

$\Rightarrow P(x,y,z) = P(x,0,z) + \beta y^2 z$

$P(x,0,z)$ is homogeneous of degree 3 in (x,z) so it is a

product of 3 linear factors; after a change of coordinates, we can assume one of the factors is $-x$:

$$P(x,y,z) = -x(x-az)(x-bz) + y^2 z \quad \text{(replace } z \text{ with } \beta z\text{)}$$

another change of coordinates: replace z with az

y with $\frac{y}{\sqrt{a}}$

$$\rightsquigarrow \quad P(x,y,z) = -x(x-z)(x-\lambda z) + y^2 z$$

<u>Note</u>: the three linear factors are <u>not</u> proportional: $(a \neq 0)$

if they were, C would be singular.

So $\lambda \in \mathbb{C} \setminus \{0, 1\}$ $\square$