MATH 150A

Elham Izadi

November 11 2016: Practice problems for second midterm

Please prove all your assertions and state all the definitions, propositions, theorems, lemmas that you use precisely. Solve the following problems by using formulas as little as possible (i.e., do things from scratch as much as you can). Good luck!

(1) Prove that if every point on a (path) connected surface M is umbilic, then M is contained in a plane or a sphere.

(2) At a certain point p on a certain parametrized surface M, we find $E = 2, F = 2, G = 4, l = 5, m = 3, n = 5$. At this point,

  (a) find the length of the vector $w = \varphi_u - \varphi_v$ in T_pM and the angle between this vector and φ_u,

  (b) find the normal curvature in the direction of w,

  (c) find the principal curvatures k_1, k_2.

(3) Let $\alpha(u)$ be a unit speed curve in $\mathbb{R}^3$, with $\kappa \neq 0$ and $\tau \neq 0$ at every point. The ruled surface M defined by $\varphi(u, v) = \alpha(u) + vT(u)$ is the tangent surface to α: it is the union of all the tangent lines at all the points of α.

  (a) At what points is M not regular? For the rest of this problem, remove these points from M.

  (b) Compute the shape operator of M, that is, $S_p(\varphi_u)$ and $S_p(\varphi_v)$, at an arbitrary point p where $v > 0$.

  (c) What are the principal curvatures and the principal directions for S_p? Find all the asymptotic directions for S_p. Is either direction φ_u or φ_v principal? Asymptotic?

(4) At a point p on a surface M, suppose that the principal curvatures satisfy $k_1 > 0 > k_2$.

  (a) Show that there are asymptotic directions at p and find a formula for the angle θ between an asymptotic direction and the principal direction u_1 which is a unit eigenvector for k_1. How many asymptotic directions are there? Why?

  (b) If the two asymptotic directions are perpendicular to each other, show that the mean curvature $H = 0$ at p.

(5) Pseudospheres: A pseudosphere is a surface of revolution similar to the sphere in that it has constant Gaussian curvature. The generating curve is of the form $\alpha(u) = (u, h(u))$ and the tangent line to α at any point reaches the x-axis after a distance of c. A curve with this property is called a *tractrix*.

(a) Show that $h' < 0$ everywhere. Write the equation of the tangent line to α at the point $(u_0, h(u_0))$.

(b) Show that

$$c = \frac{h\sqrt{1 + h'^2}}{|h'|}, \quad h' = -\frac{h}{\sqrt{c^2 - h^2}} \quad \text{and} \quad h'' = -\frac{h'c^2}{(c^2 - h^2)^{3/2}}.$$

(c) Rotate the curve α around the x-axis. Write a patch for the resulting surface of revolution and compute $\varphi_u, \varphi_v, U, E, F, G, l, m, n$ and the shape operator.

(d) Show that

$$k_\mu = \frac{h'}{c}, \quad k_\pi = -\frac{1}{ch'} \quad \text{and} \quad K = -\frac{1}{c^2}$$

where k_μ is the normal curvature along tangent directions to meridians and k_π is the normal curvature along tangent directions to parallels.