MATH 150A
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Please explain or prove all your assertions and show your work. Solve the following problems by using formulas as little as possible (i.e., do things from scratch as much as you can). Good luck!
(1) Let $M$ be the surface of equation $z=x^{2}-y^{2}$ in $\mathbb{R}^{3}$.
(a) Give a parametrization of $M$ using the variables $u=x+y, v=x-y$ as coordinates.
(b) Show that the $u$ and $v$ coordinate curves are straight lines.
(c) Explain why (b) implies that the Gaussian curvature of $M$ satistifes $K \leq 0$.
(d) Compute the angle between the $u$ and $v$ coordinate curves at the point $(u, v)=(1,1)$.
(e) Compute the Gaussian curvature of $M$ and verify that $K \leq 0$. Are there points where $K=0$ ?
(2) Let $A$ be a fixed unit vector in $\mathbb{R}^{3}$, and let $\alpha(t)$ be a curve with unit tangent vector $T(t)$. Assume that the curvature $\kappa(t)$ is nonzero everywhere. We call $\alpha$ a generalized helix with axis $A$ if the angle $\theta$ between $A$ and $T(t)$ is constant.
(a) Verify that the standard helix $\alpha(t)=(a \cos t, a \sin t, b t)$ is a generalized helix. What is the vector $A$ in this case?
(b) Show that for any generalized helix, the curvature and torsion satisfy $\tau= \pm \kappa \cot \theta$.
(3) Let $M$ be a surface in $\mathbb{R}^{3}$ and let $P$ be a plane intersecting $M$ in a curve $\alpha$. Suppose that the angle $\theta$ between $P$ and $M$ is constant along $\alpha$. Assume that $\theta \neq 0$, i.e., $P$ is not tangent to $M$.
(a) Show that $\alpha$ is a principal curve.
(b) If $\theta=\frac{\pi}{2}$, show that $\alpha$ is a geodesic.
(4) Let $M$ be a surface of revolution obtained by revolving the unit speed curve $\alpha(u)=$ $(g(u), h(u), 0)$ around the $x$-axis. Assume that $h(u)=e^{u / R}$ where $R$ is a positive constant.
(a) Show that the Gaussian curvature of $M$ is $K=-\frac{1}{R^{2}}$.
(b) Show that for every point of $M, u \leq R \ln R$.
(c) Assume that $R>1$. Compute the area of the portion of $M$ lying between the parallels $u=0$ and $u=R \ln R$.
(5) Let $M$ be the paraboloid $z=x^{2}+y^{2}$, and let $C$ be the circle $z=a^{2}$ on $M$ for some given constant $a>0$. Let $k_{n}$ be the normal curvature of $C$, i.e., the normal curvature of $M$ at any point of $C$ in the direction tangent to $C$.
(a) Explain why $k_{n}$ is the same at all the points of $C$.
(b) Compute $k_{n}$ in terms of $a$. (Make it clear which normal vector $U$ you have chosen.)
(6) At a particular point of a surface $M$ with parametrization $\varphi(u, v)$ we find $E=1, F=0, G=$ $4, l=-1, m=2, n=0$. At this point,
(a) compute the matrix of the shape operator. Is it symmetric? Should it be?
(b) From the matrix in (a), find the principal curvatures and $K$ and $H$.
(c) Find any asymptotic directions.
(d) Find the normal curvature in the direction of the vector $\varphi_{u}$.
(7) Prove that in $\mathbb{R}^{3}$ a line is the curve of least arclength between two points.
(8) (a) Prove that a curve is parametrized by arclength if and only if it has unit speed.
(b) Prove that a curve is regular if and only if it can be parametrized by arclength.
(9) Prove the Frenet formulae for a unit speed curve and from them deduce the formulae for a non unit speed curve.
(10) Let $\beta(s)$ be a unit speed curve.
(a) Prove that $\kappa=0$ if and only if $\beta$ is a line.
(b) Prove that when $\kappa>0, \tau=0$ if and only if $\beta$ is a plane curve.
(c) Prove that $\beta$ is contained in a circle if and only if $\kappa$ is constant and positive and $\tau=0$.
(11) State and prove the isoperimetric inequality.
(12) Let $M$ be a surface, $p$ a point of $M$ and $\varphi(u, v)$ a patch for $M$ around $p$. Give three equivalent definitions of the tangent space $T_{p} M$ to $M$ at $p$ and prove the equivalence of the three definitions.
(13) Let $M$ be a path connected surface in $\mathbb{R}^{3}$. Prove that if $S_{p}=0$ at every $p \in M$, then $M$ is contained in a plane.
(14) Prove that if every point on a surface $M$ is umbilic, then $M$ is contained in a plane or a sphere.
(15) (a) Give the definition of geodesic curvature and the definition of normal curvature.
(b) How are the geodesic curvature and the normal curvature related? Justify your claim.
(c) Give the definition of a geodesic.

