

14

+ Surfaces:

Unions of images of  $C^\infty$  maps

$$\underline{\varphi: D \rightarrow \mathbb{R}^3.} \quad D \subset \mathbb{R}^2 \text{ open}$$

$D$  usually an open rectangle:

$$[a, b] \times [c, d].$$

or an open disc:  $|(\mathbf{a}, \mathbf{b}) - \mathbf{P}| < r$

$D$  open: does not contain any of its boundary points.

$(u, v)$  coordinates on  $\mathbb{R}^2$ .

$$\varphi(u, v) \quad (u, v) \in D$$

$$\varphi(u, v) = (x(u, v), y(u, v), z(u, v))$$

$C^\infty$ : all partial derivatives of  $x, y, z$  exist

$$\frac{\partial x}{\partial u}$$

We can make  $D$  smaller so that:

1)  $\varphi$  is 1 to 1.

2)  $\frac{\partial \varphi}{\partial u}$  &  $\frac{\partial \varphi}{\partial v}$  are linearly independent.

$$\frac{\partial \varphi}{\partial u} = \left( \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right)$$

$$\frac{\partial \varphi}{\partial v} = \left( \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$

$$\Leftrightarrow \frac{\partial \varphi}{\partial u} \times \frac{\partial \varphi}{\partial v} \neq 0 \text{ on } D$$

↑  
normal vector to surface  
= image of  $D$  by  $\varphi$  in  $\mathbb{R}^3$ .

Ex: Graphs = Monge Patch.

$f(x, y)$  on  $D$ .

$$z = f(x, y).$$

$\varphi: D \rightarrow \mathbb{R}^3$ .

$$(x, y) \mapsto (x, y, f(x, y))$$

1-to-1:  $\varphi(x, y) = \varphi(x_1, y_1)$

$$(x, y, f(x, y)) = (x_1, y_1, f(x_1, y_1))$$

$$x = x_1 \quad y = y_1$$

$$\frac{\partial \varphi}{\partial x} = \left( 1, 0, \frac{\partial f}{\partial x} \right)$$

$$\frac{\partial \varphi}{\partial x} = (0, 1, \frac{\partial f}{\partial y}).$$

$$\begin{vmatrix} \frac{\partial \varphi}{\partial x} & \frac{\partial \varphi}{\partial y} \\ i & j \\ 0 & 0 \\ - & - \\ \end{vmatrix} = \begin{matrix} -\frac{\partial f}{\partial x} \\ + \frac{\partial f}{\partial y} \end{matrix} i - \begin{matrix} -\frac{\partial f}{\partial y} \\ -\frac{\partial f}{\partial x} \end{matrix} j.$$

$$= \left( -\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, 1 \right)$$

Normal to surface.

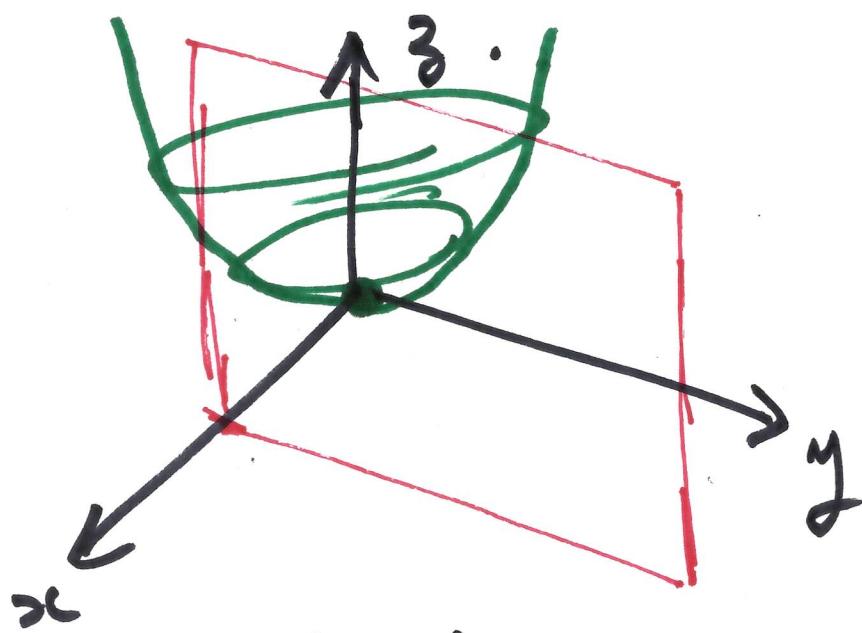
unit normal:  $\frac{1}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}} \begin{pmatrix} \frac{\partial \varphi}{\partial x} \\ \frac{\partial \varphi}{\partial y} \end{pmatrix}$

Example: the paraboloid:

$$f(x, y) = x^2 + y^2$$

$$\varphi(x, y) = (x, y, x^2 + y^2)$$

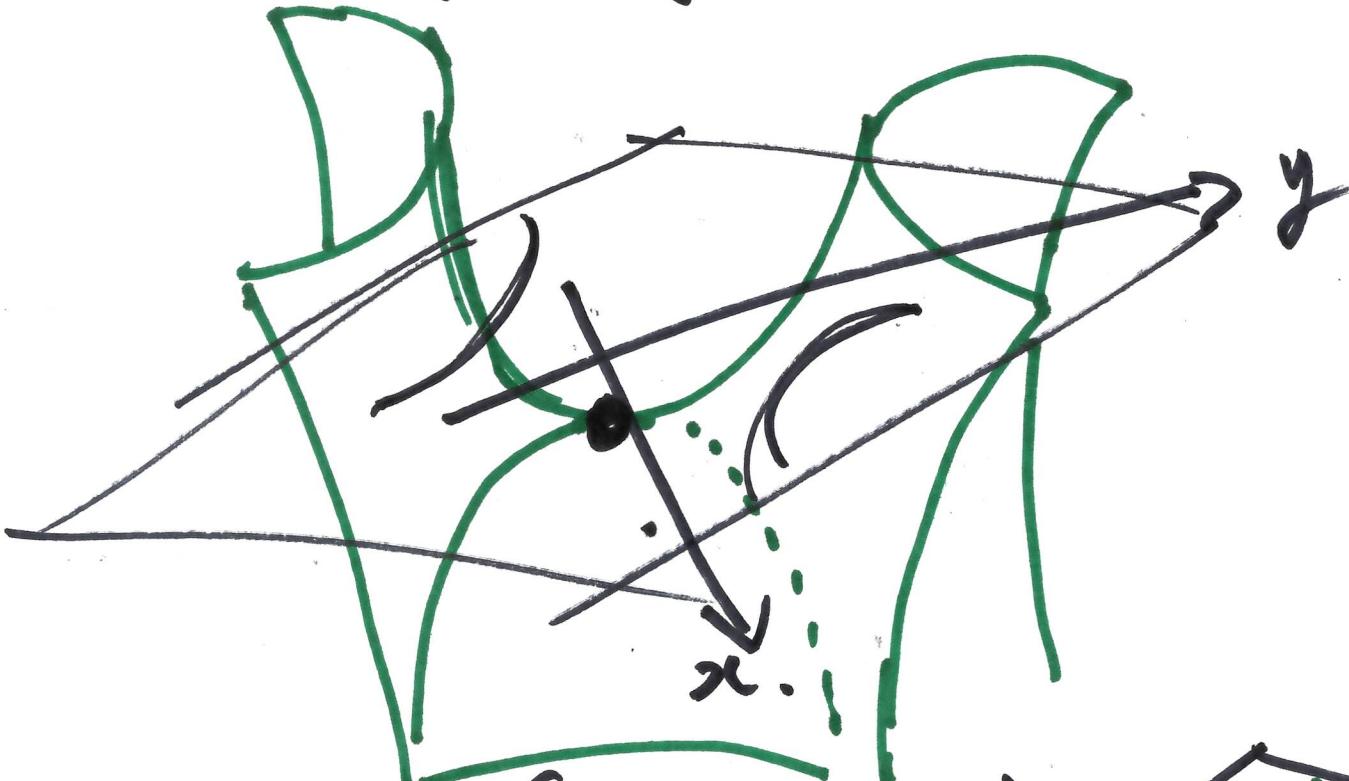
cut with  $x = 0$  : parabola.  
 $x = c$  : //



$$z = \text{constant}$$

Example: The saddle:  $f(x,y) = xy$ .

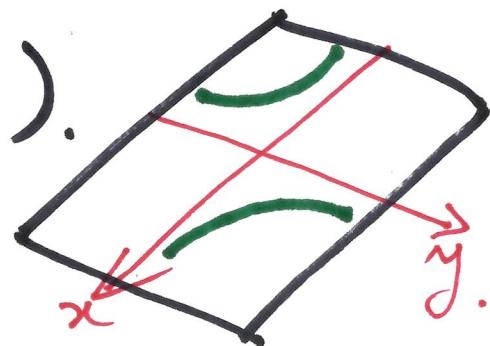
$$z = xy.$$



$$\varphi(x,y) = (x, y, xy).$$

$\exists z = \dots$

$$z = -1$$



$$x=c \quad ((c, y, cy) \text{ line})$$

$$y=c \quad (x, c, cx) \text{ line.}$$


---

Implicit equations:

$$f(x, y, z) = c.$$

if  $\boxed{z(x, y)}$  then:

$$0 = df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \quad \boxed{dz}$$

on surface.

$$\Rightarrow dz = - \frac{1}{\frac{\partial f}{\partial z}} \left( \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)$$

$$\Rightarrow \frac{\partial z}{\partial x} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

$$\frac{\partial z}{\partial y} = - \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

# Implicit function theorem

61

$$\nexists \frac{\partial f}{\partial z}(a, b, c) \neq 0 \quad \text{at } (a, b, c) \in \mathbb{R}^3$$

then.  $\exists$  neighborhood of  $(a, b, c)$   
 (= small open ball  
 with center  $(a, b, c)$ )

s.t. on this neighborhood.

$$f(x, y, z) = c \Leftrightarrow g = g(x, y).$$

$$\frac{\partial z}{\partial x} = - \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

$$\frac{\partial z}{\partial y} = - \frac{\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}$$

Ex: Sphere  $S^1$ :  
 $x^2 + y^2 + z^2 = 1$ .

$$\frac{\partial f}{\partial z} = 2z$$

$$z = \sqrt{1-x^2-y^2} = f(x, y).$$

$$\varphi: D \rightarrow \mathbb{R}^3$$

$\subset \mathbb{R}^2$

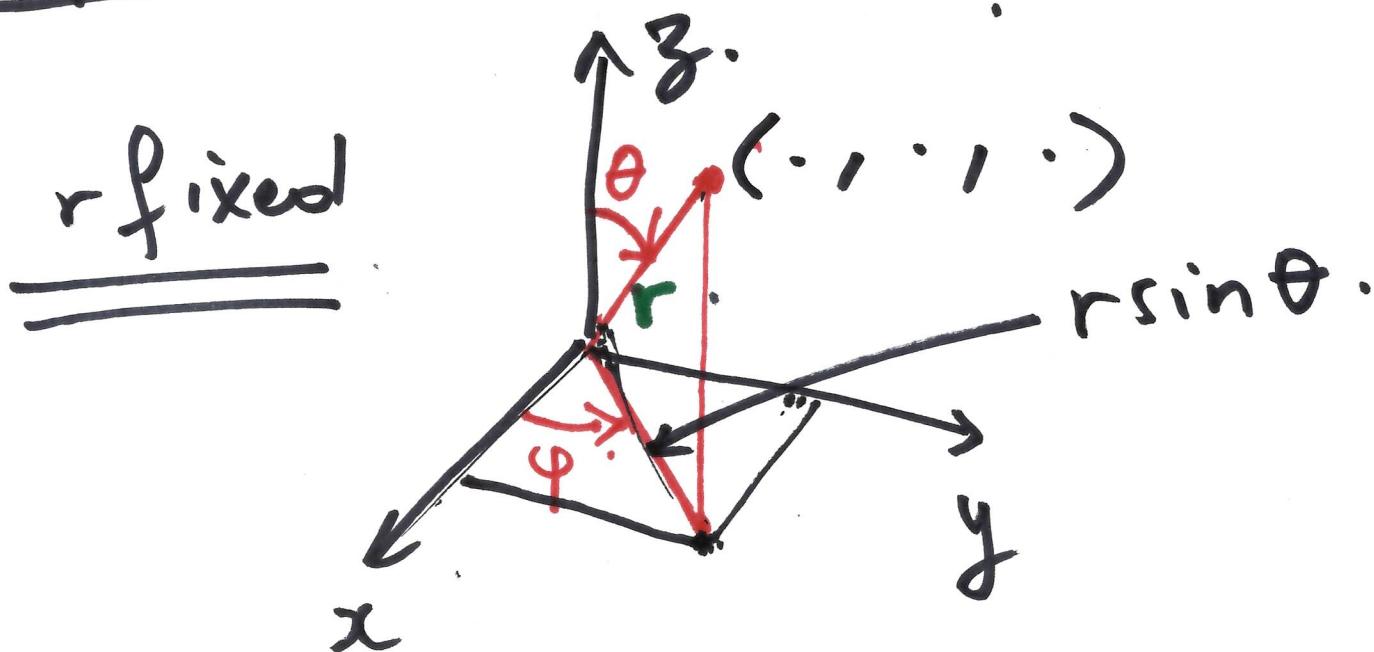
$$(x, y) \mapsto (x, y, \sqrt{1-x^2-y^2})$$

$D = \text{open disc: } \{x^2+y^2 < 1\}$

image = upper hemisphere..

---

### Spherical coordinates:



$$(r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$$

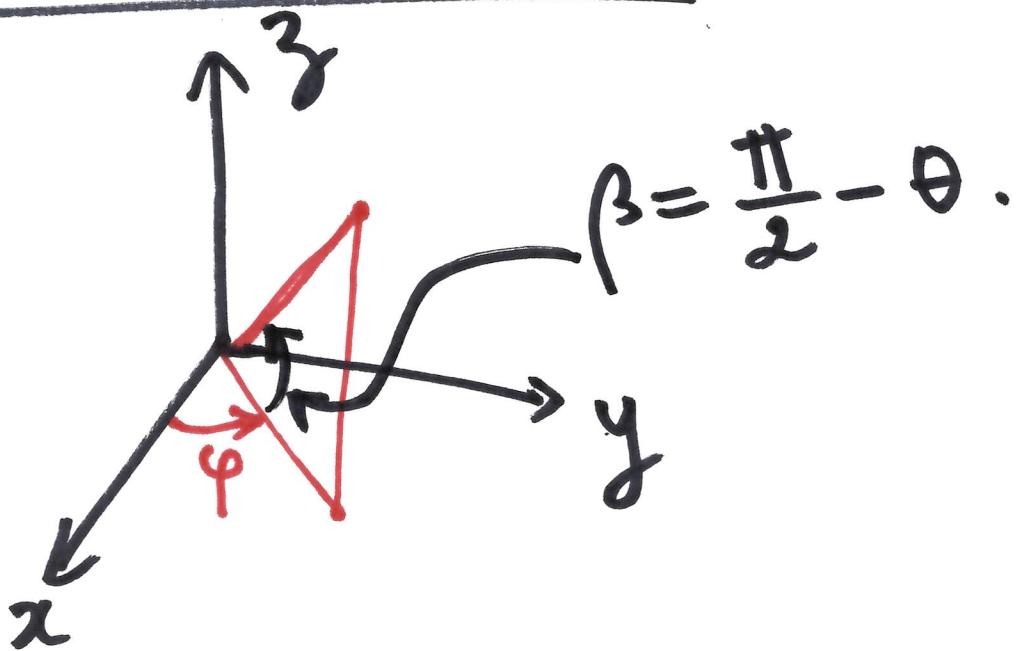
$0 < \varphi < \pi$        $0 < \theta < \pi$ .  
 right hemisphere.

$$\text{allow } 0 < \varphi < 2\pi \quad 0 < \theta < \pi^{18}$$

whole sphere except a ~~half~~<sup>18</sup> half circle.

---

Geographic coordinates:



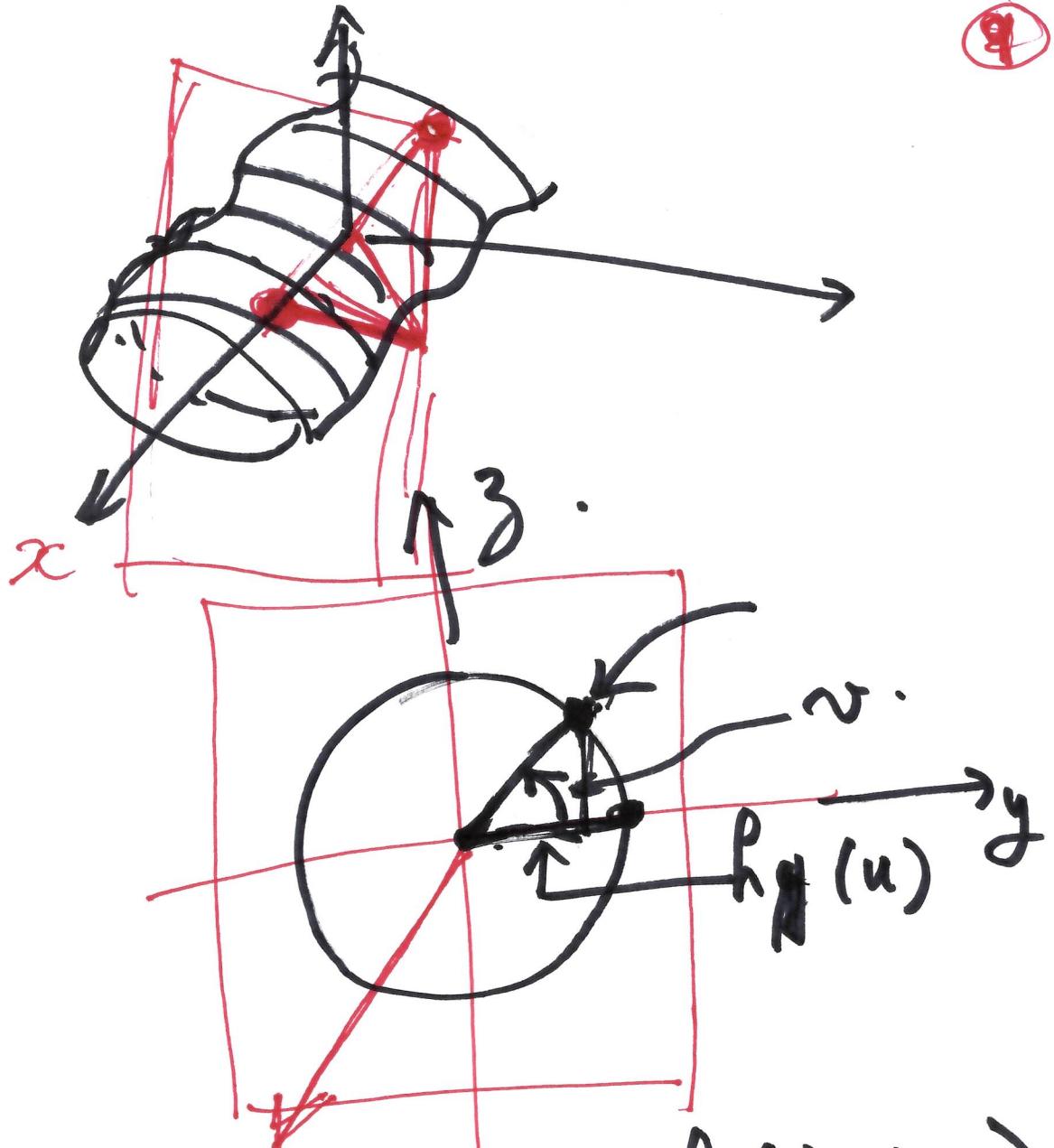
Surfaces of revolution:

$$\text{curve } \alpha(\bar{t}) = (g(\bar{u}), 0)$$

$$\alpha(u) = (f(u), g(u), 0)$$

rotate around x-axis:

$$(g(u), f(u), 0)$$



$(g(u), h(u) \cos v, h(u) \sin v)$

$D = \boxed{\text{domain of } \alpha} \times ]0, 2\pi[.$

$\alpha: I \rightarrow \mathbb{R}^3$ .

$\alpha(u) = (g(u), h(u), 0)$

$D = I \times ]0, 2\pi[.$

Might need to make  $I$  smaller to make  $\varphi$  1 to 1. (10)

$$\varphi(u, v) = (g(u), h(u)\cos v, h(u)\sin v)$$

$$\frac{\partial \varphi}{\partial u} = (g', h'\cos v, h'\sin v)$$

$$\frac{\partial \varphi}{\partial v} = (0, -h^2 \sin v, h^2 \cos v)$$

$$\begin{vmatrix} i & j & k \\ g' & h'\cos v & h'\sin v \\ 0 & -h\sin v & h\cos v \end{vmatrix} = -g'h\cos v j - g'h\sin v k.$$

$$\text{length} = \sqrt{h^2 h'^2 + g'^2 h^2}$$

$$= |h| \sqrt{g'^2 + h'^2}$$

$\neq 0$  if  $\underline{h \neq 0}$  &  $\underline{\alpha \text{ regular}}$

Tangent planes:  $M \ni P$  (11)

(1)  $T_P M = \text{union of all tangent lines to } M \text{ at } P$   
curves in

(2) patch  $\varphi: D \rightarrow \mathbb{R}^3$ .

$P = \text{image of } (\varphi_u, \varphi_v) \in D$

$\varphi = (x(u, v), y(u, v), z(u, v))$

$\varphi_u \quad \varphi_v.$

plane through  $P$   
it contains  $\varphi_u(u_0, v_0) \quad \varphi_v(u_0, v_0)$

(3) Plane through  $P$  with  
normal vector  $\varphi_u(u_0, v_0) \times \varphi_v(u_0, v_0)$

---

$\alpha: I \rightarrow \mathbb{R}^3$

$\gamma$  image of  $\alpha \subset M$ .

we say the curve lies in  $M$ .

$\varphi: D \rightarrow M \subset \mathbb{R}^3$ .

$$\alpha : I \xrightarrow{\alpha} M \xleftarrow{\varphi} D$$

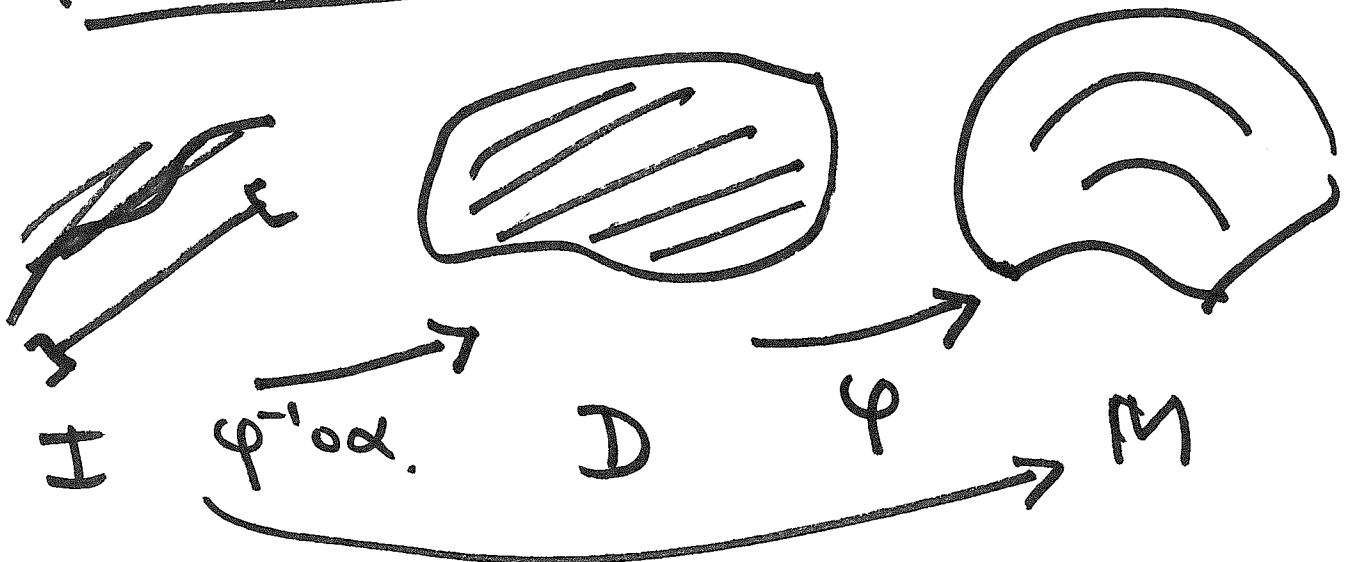
(12)

$$\varphi^{-1}\alpha : I \rightarrow D$$

$$\varphi^{-1}\alpha(t) = (u(t), v(t))$$

$C^\infty$ .

$$\begin{aligned} \alpha &= \varphi_0 \varphi^{-1} \circ \alpha = \varphi(\varphi^{-1}(\alpha)) \\ \boxed{\alpha} &= \varphi(u(t), v(t)) \end{aligned}$$



$\alpha'$  = tangent vector to curve.

Assume  $u(0) = u_0$   
 $v(0) = v_0$ .

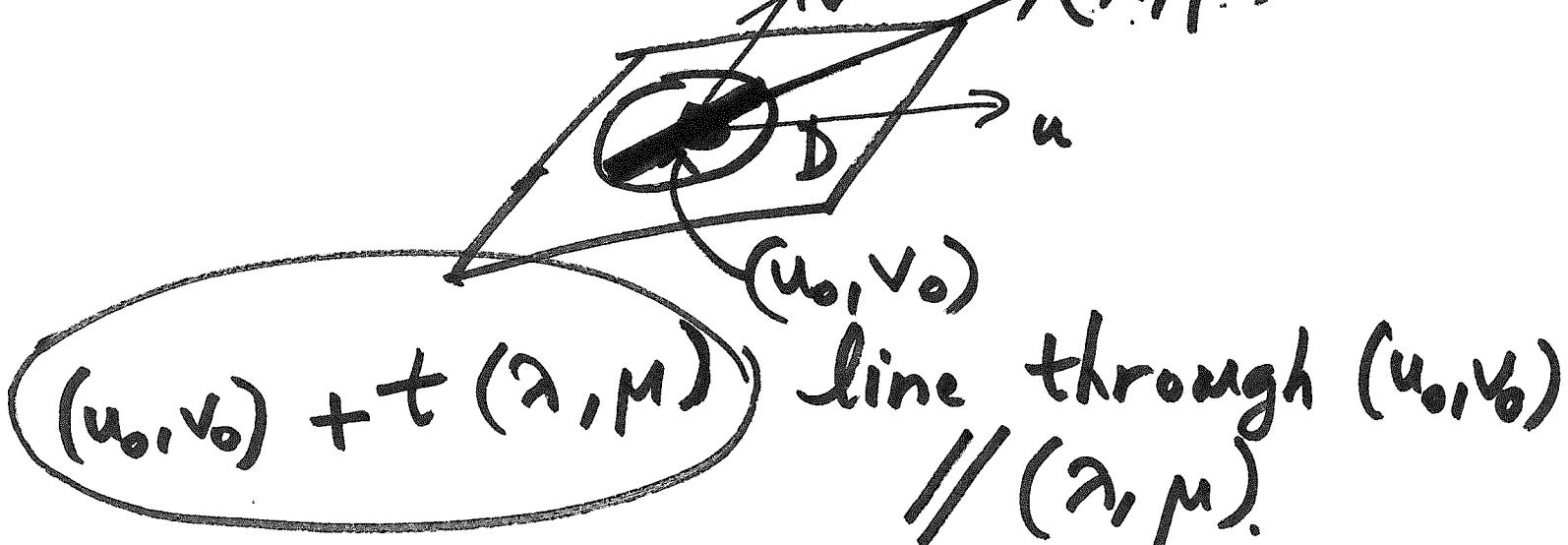
or  $u(t_0) = u_0$   
 $v(t_0) = v_0$

$$\begin{aligned}
 \alpha'_{t_0} &= (\varphi(u(t), v(t)))' \\
 &= \varphi_u \boxed{u'(t_0)}_{u_0, v_0} + \varphi_v \boxed{v'(t_0)}_{u_0, v_0} \\
 &= \underline{\underline{u'(t_0)}} \varphi_u + \underline{\underline{v'(t_0)}} \varphi_v.
 \end{aligned}$$

So  $\alpha'(t_0) \in \langle \varphi_u, \varphi_v \rangle$  at P.

Conversely, suppose we have

$$\lambda \varphi_u + \mu \varphi_v \text{ at } (P)$$



$(u_0, v_0) + t(\lambda, \mu)$  line through  $(u_0, v_0)$   
 $\parallel (\lambda, \mu)$ .

$\alpha := \varphi(u_0 + t\lambda, v_0 + t\mu)$   
 $t$  small enough.

$$(u_0, v_0) + t(\lambda, \mu) \in D$$

tangent vector:

$$\varphi_u|_{(u_0, v_0)} \cdot \lambda + \varphi_v|_{(u_0, v_0)} \cdot \mu$$

$$D_U := \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}$$

Def:

We say  $M$  is orientable if  
 $\exists$  a consistent normal  
vector field on  $M$ .



From now on we assume  $M$   
orientable. can choose an

14

orientation.

Oriented: we have already chosen one.

We also assume  $M$  is path connected. Any two points can be joined by a curve.

$P, Q \in M, \exists \alpha: [0, 1] \rightarrow M$

s.t.  $\alpha(0) = P, \alpha(1) = Q$ .

Def: The shape operator

$P \in M. V \in T_P M$

$$S_P(V) := -\nabla_V V$$

After choosing a unit normal vector field ( $M$  oriented)

$U = (f, g, h)$  function

$V = (a, b, c)$  constants

$$\nabla_{\sqrt{t}} v = (\nabla_v f, \nabla_v g, \nabla_v h) \quad (16)$$

$$f(x_1, y, z) \quad g(x_1, y, z) \quad h(x_1, y, z)$$

$$\nabla_v f = a \frac{\partial f}{\partial x} + b \frac{\partial f}{\partial y} + c \frac{\partial f}{\partial z}.$$

Theorem:

$$S_p : T_p M \rightarrow T_p M$$

is linear

Proof: more  $S_p(v) \in T_p M$

$$\forall v \in T_p M.$$

$$S_p(v) \in T_p M \Leftrightarrow v \cdot S_p(v) = 0$$

$$\Leftrightarrow -v \cdot \nabla_{\sqrt{t}} v = \cancel{(\cancel{\frac{1}{2} \nabla v})}$$

$$= -\frac{1}{2} (\nabla_{\sqrt{t}} (v \cdot v))$$

$$= -\frac{1}{2} \nabla_{\sqrt{t}} (1) = 0$$

linear means:

$$\forall \lambda \in \mathbb{R} \quad \forall v \in T_p M \quad (17)$$

$$S_p(\lambda v) = \lambda S_p(v)$$

$$\forall v, w \in T_p M$$

$$S_p(v+w) = S_p(v) + S_p(w).$$


---

Theorem: If  $S_p \neq 0 \quad \forall p$   
 (i.e.,  $S_p(v) = 0 \quad \forall v$ )

then  $M$  is contained in a plane.

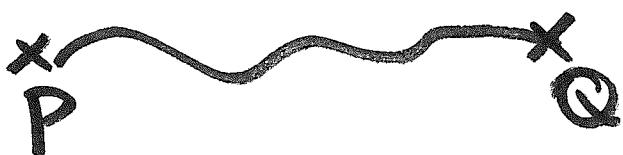
Proof: Choose  $P \in M$ .

$$Q \in T_p M \Rightarrow (Q-P) \cdot V(P) = 0$$

Choose  $Q \in M$ .

$$\exists \alpha : [0, 1] \rightarrow M$$

$$\text{s.t. } \alpha(0) = P, \alpha(1) = Q.$$



$$(\alpha(t) - \beta) \cdot V(\alpha(t)) = f(t) \quad (18)$$

derivative:

$$f'(t) = \alpha'(t) \cdot V(\alpha(t))$$

$$+ (\alpha(t) - \beta) \cdot (V(\alpha(t)))'$$

$$V(\alpha(t))' = \frac{\partial V}{\partial x} \cdot (x' + \frac{\partial V}{\partial y} y' + \frac{\partial V}{\partial z} z')$$

$$\alpha(t) = (x(t), y(t), z(t))$$

$$\hookrightarrow = \underset{\alpha'(t)}{\nabla} V(\alpha(t)) = -S_p(\alpha'(t))$$

$\stackrel{=0}{\text{by}}$   
assumption.

$$\alpha'(t) \cdot V(\alpha(t)) = 0$$

because  $\alpha'(t) \in T_{\alpha(t)} M$

$$\Rightarrow f' = 0 \quad \forall t.$$

$\Rightarrow f$  is constant.

$$\forall t \quad f(t) = f(0) = f_0$$

(19)

$$\cancel{(Q-P) \cdot U(Q(0)) = 0}$$

$$f(0) = 0 \quad f(1) = (P-Q) \cdot U(P)$$

□

Example: Sphere of radius R.  
centered at 0.

patch  $\varphi(u, v) =$

$$R(\cos u \cos v, \sin u \cos v, \sin v)$$

geographic

$$\varphi_u = (-\sin u \cos v, \cos u \cos v, 0)$$

$$\varphi_v = (-\cos u \sin v + \sin u \cos v, \sin u \sin v, \cos v)$$

$$\begin{vmatrix} i & j & k \\ -\sin u \cos v & \cos u \cos v & 0 \\ -\cos u \sin v & -\sin u \sin v & \cos v \end{vmatrix}$$

$$i(\cos u \cos^2 v) - j(-\sin u \cos v)$$

$$+ k(\sin^2 u \sin v \cos v + \cos^2 u \sin v \cos v)$$

$$= (\cos u \cos^2 v, \sin u \cos^2 v, \sin v \cos v) \quad (20)$$

length:

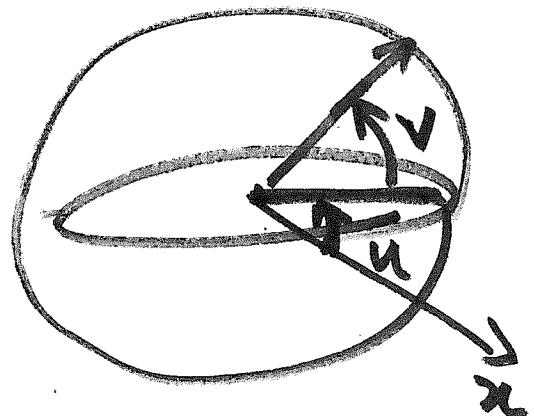
$$\sqrt{\cos^2 u \cos^4 v + \sin^2 u \cos^4 v + \sin^2 u \cos^2 v}$$

$$\sqrt{\cos^4 v + \sin^2 v \cos^2 v}$$

$$\sqrt{\cos^2 v}$$

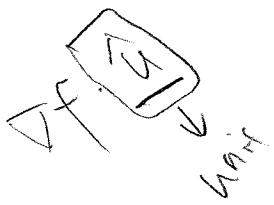
$$= |\cos v|$$

$$= \cos v.$$



$$v = \frac{1}{\cos v} (-\dots)$$

$$= (\cos u \cos v, \sin u \cos v, \sin v)$$



unit

$$\varphi(u, v) = (u \cos v, u \sin v, bv)$$

$b \neq 0$

$$D = ? \quad \mathbb{R}^2$$

where is it  $I - \text{to- } 1?$

$$(u_1 \cos v_1, u_1 \sin v_1, bv_1)$$

$$= (u_2 \cos v_2, u_2 \sin v_2, bv_2)$$

$$\Rightarrow bv_1 = bv_2 \quad b \neq 0$$

$$\Rightarrow v_1 = v_2.$$

$$\Rightarrow \omega_1 v_1 = \omega_2 v_2 \quad \& \sin v_1 = \sin v_2.$$

$$u_1 \cos v_1 = u_2 \cos v_1$$

$$u_1 \sin v_1 = u_2 \sin v_1$$

$\sin v_1$  &  $\cos v_1$  are never simultaneously

$$0 \quad \text{so} \quad u_1 = u_2$$

$$\varphi_u = (\cos v, \sin v, 0)$$

$$\varphi_v = (-u \sin v, u \cos v, b)$$

$$\varphi_u \times \varphi_v :$$

$$\begin{vmatrix} i & j & k \\ \cos v & \sin v & 0 \\ -u \sin v & u \cos v & b \end{vmatrix} = i(b \sin v) - j(b \cos v) + uk(\cos^2 v + \sin^2 v)$$

$$= (b \sin v, -b \cos v, u)$$

never zero because sine & cosine are never simultaneously 0.

$$\text{Domain} = \mathbb{R}^2$$

tangent plane at a point

$$(u_0 \cos v_0, u_0 \sin v_0, b v_0) = P_0$$

$(P - P_0) \cdot U = 0$ , equation of the plane  $\perp$  to  $U$  & containing  $P_0$

$$|P_u \times P_v| = \sqrt{b^2 \sin^2 v + b^2 \cos^2 v + u^2}$$

$$= \sqrt{b^2 + u^2}$$

$$V = \frac{1}{\sqrt{b^2 + u^2}} (b \sin v, -b \cos v, u)$$

equation of tangent plane at  $P_0$ :

$$P = (x, y, z)$$

$$(x - u_0 \cos v_0, y - u_0 \sin v_0, z - bv_0) \cdot V_{P_0} = 0$$

or

$$(x - u_0 \cos v_0) b \sin v_0 + \\ + (y - u_0 \sin v_0) (-b \cos v_0) + \\ + (z - bv_0) u_0 = 0$$

$$b \sin v_0 [x - b \cos v_0 y] + u_0 [z] = \\ \underline{bu_0 \cos v_0 \sin v_0 - bu_0 \sin v_0 \cos v_0 + bv_0 u_0}$$

$$\nabla \varphi(u, v) = (u \cos v, u \sin v, bv)$$

$$\underline{u = u_0}$$

$$\varphi(u_0, v) = (u_0 \cos v, u_0 \sin v, bv)$$

helix

$$v = v_0$$

$$\varphi(u, v_0) = (u \cos v_0, u \sin v_0, bv_0)$$

$$x = u \cos v, \quad y = u \sin v, \quad z = bv$$

$$x^2 + y^2 = u^2$$

$$\underline{y = x \tan v}$$

$$\frac{y}{x} = \tan v. = \tan\left(\frac{\beta}{b}\right)$$

$$\frac{y}{x} - \tan\left(\frac{\beta}{b}\right) = 0$$

$$\underbrace{g(x, y, z)}_{g}$$

$$y - x \tan\left(\frac{\beta}{b}\right) = 0$$

$$\frac{\partial g}{\partial y} = 1 \neq 0 \quad \exists \text{ function } f(x, z)$$

near every point.

s.t.  $y = f(x, z)$  on the surface.

By the implicit function theorem

$$y = x \tan\left(\frac{\beta}{b}\right) = f(x, z)$$

Monge patch:

$$\psi(x, z) = \left( x, x \tan\left(\frac{z}{b}\right), z \right)$$

$$P \in M \quad P = (u \cos v, u \sin v, bv)$$

$$S_p(v) = -\nabla_v v.$$

$$S_p(\varphi_u) = -\nabla_{\varphi_u} v = -$$

$$v = (v_1, v_2, v_3)$$

$$\nabla_{\varphi_u} = (\cancel{\nabla_{\varphi_u} v_1}, \nabla_{\varphi_u} v_2, \nabla_{\varphi_u} v_3)$$

$$\nabla_{\varphi_u} v_1 = (\varphi_u)_1 \cdot \frac{\partial v_1}{\partial x} + (\varphi_u)_2 \frac{\partial v_1}{\partial y}$$

$$+ (\varphi_u)_3 \cdot \frac{\partial v_1}{\partial z}$$

$$= \left( \frac{\partial \varphi}{\partial u} \right)_1 \frac{\partial v_1}{\partial x} + \left( \frac{\partial \varphi}{\partial u} \right)_2 \frac{\partial v_1}{\partial y}$$

$$+ \left( \frac{\partial \varphi}{\partial u} \right)_3 \frac{\partial v_1}{\partial z}$$

$$= \frac{\partial v_1}{\partial u}$$

$$v_1(x_1, y_1, z) = \cancel{H(\varphi(x_1, y_1, z))} \\ = v_1(\varphi(u, v)).$$

$$\Rightarrow \frac{\partial v_1}{\partial u} = \frac{\partial v_1}{\partial x} \cdot \left( \frac{\partial \varphi}{\partial u} \right)_1 + \frac{\partial v_1}{\partial y} \cdot \left( \frac{\partial \varphi}{\partial u} \right)_2 \\ + \frac{\partial v_1}{\partial z} \cdot \left( \frac{\partial \varphi}{\partial u} \right)_3 \\ = \nabla v_1 \cdot \varphi_u$$

~~$$\nabla_{\varphi_u} v = \frac{\partial v}{\partial u}$$~~

$$= \frac{\partial}{\partial u} \left( \frac{1}{\sqrt{u^2 + b^2}} (b \sin v, -b \cos v, u) \right)$$

$$= \frac{\partial}{\partial u} \left( (u^2 + b^2)^{-1/2} (b \sin v, -b \cos v, u) \right)$$

$$= -\frac{1}{2} \frac{2u}{(\sqrt{u^2 + b^2})^3} (b \sin v, -b \cos v, u)$$

$$+ \frac{1}{\sqrt{u^2 + b^2}} (0, 0, 1)$$

$$\nabla_{\varphi_v} V = \frac{\partial V}{\partial v}$$

$$= \frac{1}{\sqrt{u^2+b^2}} (b \cos v, + b \sin v, 0)$$

$$= \frac{b}{\sqrt{u^2+b^2}} \varphi_u$$

$$\nabla_{\varphi_u} V = \frac{\mu}{(\sqrt{u^2+b^2})^3}$$

$$= \frac{1}{(\sqrt{u^2+b^2})^3} \left( -ub \sin v, bu \cos v, -u^2+u^2+b^2 \right)$$

$$= \frac{1}{(\sqrt{u^2+b^2})^3} (-bu \sin v, bu \cos v, b^2)$$

$$= \frac{b}{(\sqrt{u^2+b^2})^3} \varphi_v$$

$$\Delta_{\varphi_{uv}} \psi = \frac{b}{(\sqrt{u^2 + b^2})^3} \varphi_v + o \varphi_u$$

$$\nabla_{\varphi_v} \psi = \frac{b}{\sqrt{u^2 + b^2}} \varphi_u + o \varphi_v$$

$$S_p = - \begin{pmatrix} 0 & & & \\ & \frac{b}{(\sqrt{-})^3} & & \\ & & 0 & \\ & & & b \end{pmatrix}$$

$$\theta(u) = \int_0^u \kappa(t) dt$$

$$\theta(\pm) = \int_{\pm}^u \frac{1}{t} dt = \ln u$$

$$\beta(s) = \left( \int_1^s \cos \theta(u) du, \int_1^s \sin \theta(u) du \right)$$

compute  $\kappa$  for

$$\beta(s) = \left( \int_0^s \cos \theta(u) du, \int_0^s \sin \theta(u) du \right)$$

$$\kappa = \frac{|x'y'' - x''y'|}{(x'^2 + y'^2)^{3/2}} = 1$$

$$= |x'y'' - x''y'|$$

$$= |\cos \theta(s) \cos \theta'(s) \cdot \frac{d\theta}{ds}$$

$$+ \sin \theta(s) \sin \theta'(s) \cdot \frac{d\theta}{ds}|$$

$$= \left| \frac{d\theta}{ds} \right|$$

$$\delta(s) = \int_0^s \kappa(t) dt$$

$$= \kappa(s)$$

$$\alpha(t) \quad T(t) = \frac{\alpha'(t)}{|\alpha'(t)|}$$

$$\alpha'(t) \neq 0$$

$$|\alpha'(t)| = v(t) = \boxed{\frac{ds}{dt}}$$

$$\alpha'(t) = v(t) T(t)$$

$$\alpha''(t) = v'(t)T(t) + v(t)T'(t)$$

$$\alpha' \parallel \alpha'' \Leftrightarrow T \parallel T'$$

$$T \cdot T^* = 1 \Rightarrow \alpha T \cdot T' = 0$$

$$\Rightarrow T \perp T'$$

$$\Rightarrow T' = 0 \quad \left| \frac{dT}{ds} \right| = K(s)$$

$$T' = \frac{dT}{dt} = \frac{dT}{ds} \cdot \frac{ds}{dt}$$

$$T' = 0 \Rightarrow \frac{dT}{ds} = 0$$

$\Rightarrow K = 0 \Rightarrow$  curve is a line.

# The Linear Algebra of Surfaces:

Review of linear algebra:

$2 \times 2$  matrices:

linear transformations:  $\mathbb{R}^2 \rightarrow \mathbb{R}^2$

$T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$   $\{\tilde{v}_1, \tilde{v}_2\}$  basis of  $\mathbb{R}^2$ .

$$T(\tilde{v}_1) = \gamma_{11}\tilde{v}_1 + \gamma_{12}\tilde{v}_2.$$

$$T(\tilde{v}_2) = \gamma_{21}\tilde{v}_1 + \gamma_{22}\tilde{v}_2.$$

$$\gamma_{ij} \in \mathbb{R}$$

matrix of  $T$  on  $\{\tilde{v}_1, \tilde{v}_2\}$ :

$$\begin{array}{c} \tilde{v}_1 \\ \tilde{v}_2 \end{array} \longrightarrow \left( \begin{array}{c|c} \gamma_{11} & \gamma_{12} \\ \hline \gamma_{21} & \gamma_{22} \end{array} \right)$$

$$T(\tilde{v}_1) \quad T(\tilde{v}_2)$$

Eigen vectors & Eigenvalues:

We say that  $\lambda \in \mathbb{R}$  is an

eigenvalue for  $T$  if  $\exists v \neq 0$

s.t.  $T(v) = \lambda v$

$v$  is called an eigenvector for  $\lambda$ .

If we can find a basis  $\{v_1, v_2\}$   
s.t.  $v_1, v_2$  are both eigenvectors,  
then  $T(v_1) = \lambda_1 v_1$ ,

$$T(v_2) = \lambda_2 v_2$$

matrix of  $T$ : 
$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

We say we have diagonalized  $T$ .

$\det T = \det$  of any matrix w.r.t  $T$

$\text{trace } T = \text{trace } //$

$$\det T = \lambda_1 \lambda_2$$

$$\text{trace } T = \lambda_1 + \lambda_2.$$

$$\underline{\text{Proof:}} \quad \varphi_u \cdot U = 0$$

$$0 = \frac{\partial}{\partial u} (\varphi_u \cdot U) = \varphi_{uu} \cdot U + \varphi_u \cdot \frac{\partial U}{\partial u}$$

$$S_F(U) = -\nabla_U U \quad \frac{\partial U}{\partial u} = -S_F(\varphi_u)$$

$$\text{So } S_F(\varphi_u) \cdot \varphi_u = -\varphi_u \cdot \frac{\partial U}{\partial u} \\ = \varphi_{uu} \cdot U$$

$$0 = \frac{\partial}{\partial v} (\varphi_u \cdot U) = \varphi_{uv} \cdot U + \varphi_u \cdot \frac{\partial U}{\partial v}$$

$$\frac{\partial U}{\partial v} = -S_F(\varphi_v)$$

$$S_F(\varphi_v) \cdot \varphi_u = -\varphi_u \cdot \frac{\partial U}{\partial v} \\ = \varphi_{uv} \cdot U$$

$S_F$  symmetrici  $\Leftrightarrow w_1, w_2 \in T_p M$

show  $S_F(w_1) \cdot w_2 = w_1 \cdot S_F(w_2)$

$$w_1 = a \varphi_u + b \varphi_v$$

$$w_2 = c \varphi_u + d \varphi_v$$

$$S_F(w_1) = S_F(a \varphi_u + b \varphi_v)$$

Def: We say  $T$  is symmetric if  
 $\forall v, w \in \mathbb{R}^2$

$$T(v) \cdot w = v \cdot T(w)$$

Ex. 2.3.4: With respect to an orthonormal basis, the matrix of a symmetric transformation is symmetric.

orthonormal means all basis vectors have length 1 and are orthogonal = perpendicular.

$$\varphi: D \rightarrow \mathbb{R}^3$$

Theorem (2.3.5)  $M$  surface  $p \in M$

$S_p$  is symmetric :  $T_p M \xrightarrow{\text{SI}} \mathbb{R}^2 \xrightarrow{\text{II}} \mathbb{R}^2$

Also :  $S_p(\varphi_u) \cdot \varphi_u = \varphi_{uu} \cdot v$

$S_p(\varphi_u) \cdot \varphi_v = \varphi_{uv} \cdot v = S_p(\varphi_v) \cdot \varphi_u$

$S_p(\varphi_v) \cdot \varphi_v = \varphi_{vv} \cdot v$

$$= a \Sigma_p(\varphi_u) + b \Sigma_p(\varphi_v)$$

$$\Sigma_p(w_1) \cdot w_2 = (a \Sigma_p(\varphi_u) + b \Sigma_p(\varphi_v)) \cdot w_2$$

$$= a \Sigma_p \Sigma_p(\varphi_u) \cdot w_2 + b \Sigma_p(\varphi_v) \cdot w_2$$

$$= a \Sigma_p(\varphi_u) \cdot (c \varphi_u + d \varphi_v) + b \Sigma_p(\varphi_v) \cdot (c \varphi_u + d \varphi_v)$$

$$= ac \Sigma_p(\varphi_u) \cdot \varphi_u + ad \Sigma_p(\varphi_u) \cdot \varphi_v$$

$$+ bc \Sigma_p(\varphi_v) \cdot \varphi_u + bd \Sigma_p(\varphi_v) \cdot \varphi_v$$

Similarly  $w_1 \cdot \Sigma_p(w_2) = \Sigma_p(w_2) \cdot w_1$

$$= ac \Sigma_p(\varphi_u) \cdot \varphi_u + bc \Sigma_p(\varphi_u) \cdot \varphi_v$$

$$+ ad \Sigma_p(\varphi_v) \cdot \varphi_u + bd \Sigma_p(\varphi_v) \cdot \varphi_v$$

we know  $\Sigma_p(\varphi_u) \cdot \varphi_v = \varphi_u \cdot \Sigma_p(\varphi_v)$

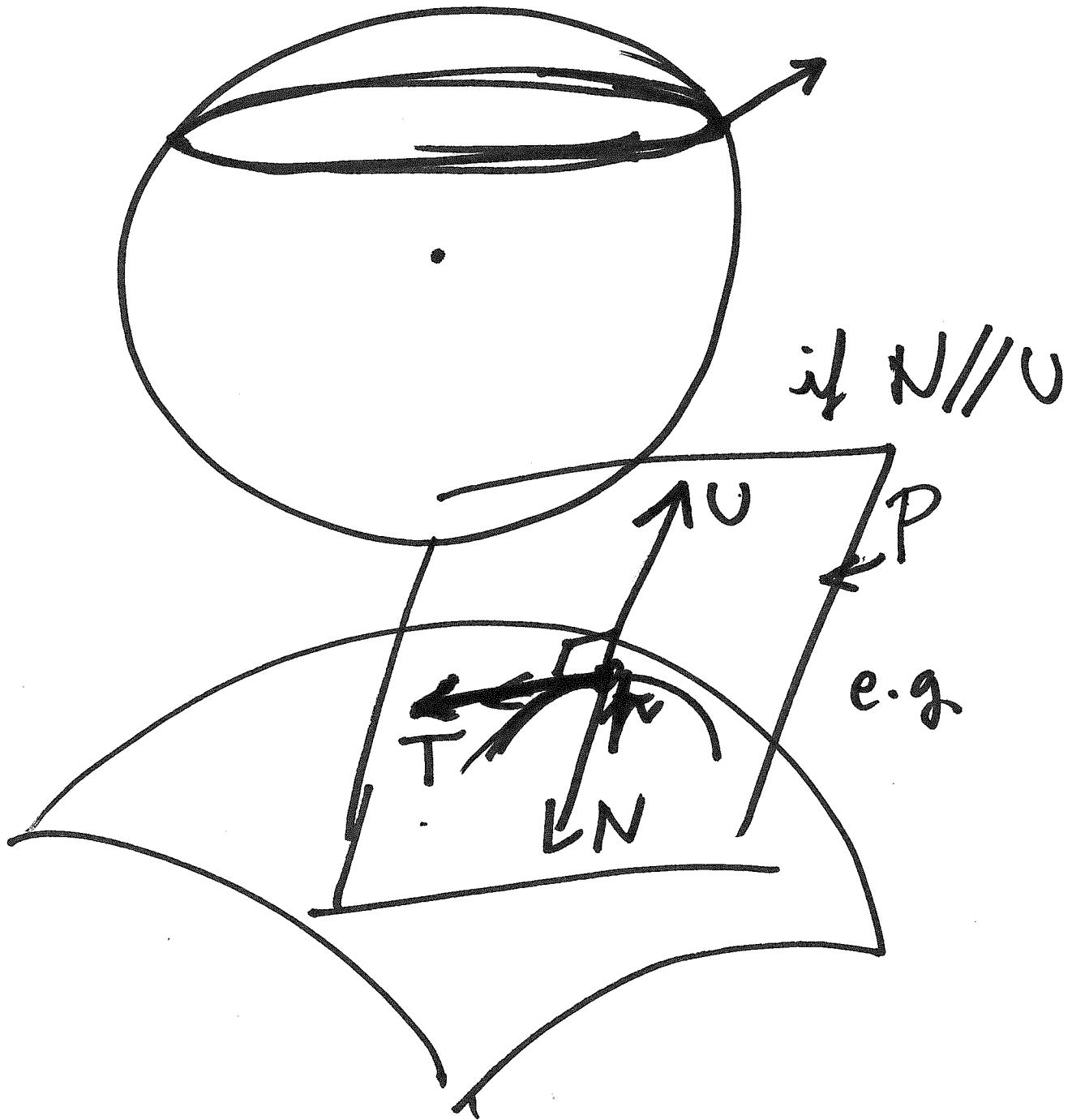
Lemma: (2.4.1) For any curve  $\alpha(t)$

in  $M$

$$\alpha'' \cdot v = S(\alpha') \cdot \alpha'$$

$$\alpha'' \cdot v = |\alpha''| |v| \cos \theta$$

$$= \kappa \cos \theta.$$



Prop. (2.4.3) if  $\gamma$  a unit vector.  
 P the plane through  $\gamma$  and  
 parallel to  $\gamma$  and  $v$

Proof: (3rd proof in book)

$$\alpha' \cdot v = 0$$

$$0 = \frac{d}{dt} (\alpha' \cdot v) = (\alpha' \cdot v)'$$

$$= \alpha'' \cdot v + \alpha' \cdot \frac{dv}{dt}$$

$$= \alpha'' \cdot v \Rightarrow -\alpha' \cdot S_p(\alpha')$$

$$\Rightarrow \alpha'' \cdot v = S_p(\alpha') \cdot \alpha' \quad \square$$

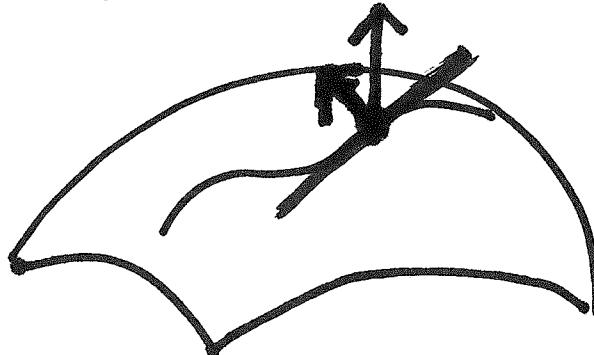
$\alpha'' \cdot v$  = component of acceleration  
in direction of  $v$

It only depends on  $\alpha'$

For a unit speed curve it only  
depends on the tangent direction

$$\alpha' = T$$

$$\alpha'' = \kappa N$$



$$N \angle \theta \leftarrow U$$

$\sigma = \sigma =$  the curve  $P \cap M$

parametrized by arc length with

$$\sigma(0) = p \text{ and } \sigma'(0) = \gamma = T$$

Then the curvature of  $\sigma$  at 0

$$\text{is } \pm \sigma''. \nu \left( = S_p(\sigma') \cdot \sigma' \right)$$
$$= S_p(\gamma) \cdot \gamma$$

Proof:  $\sigma'' \cdot \nu = |\sigma''| |\nu| \cos \theta$

$$N \parallel \nu \Rightarrow \theta = 0 \text{ or } \pi$$

$$= \pm |\sigma''|$$

$$= \pm K_\sigma(0)$$

---

$$K_\sigma(0) = \pm \sigma''(0) \cdot \nu$$

$$= \pm S_p(\sigma'(0)) \cdot \sigma'(0)$$

$$= \pm S_p(\gamma) \cdot \gamma$$

Definition: (1) Given a unit vector

$\gamma$  in  $T_p M$ , the curvature of  $M$  in the direction of  $\gamma$  is

$$k(\gamma) := S_p(\gamma) \cdot \gamma$$

---

(2) The (real) eigenvalues of  $S_p$  are the principal curvatures of  $M$  at  $p$  : denoted  $k_1, k_2$ .

the corresponding eigenvectors are the principal directions at  $p$ .

(3) The determinant

$K := k_1 k_2$  is the Gaussian curvature

(4) Half the trace

$H := \frac{1}{2} (k_1 + k_2)$  is the

mean curvature

(5) A point where  $k_1 = k_2$  is called umbilic.

$$\gamma \quad S_p(\gamma) \cdot \gamma.$$

If  $\gamma$  is an eigenvector, then

$$\exists \lambda \text{ s.t. } S_p(\gamma) = \lambda \gamma.$$

then  $k(\gamma) = S_p(\gamma) \cdot \gamma$

$$= \lambda \gamma \cdot \gamma = \lambda$$

is  $\lambda$  a principal curvature.

---

Recall: Choice of orientation = choice of unit normal vector field

Change orientation:  $U \rightsquigarrow -U$

$$S \rightsquigarrow -S$$

$$k_i \rightsquigarrow -k_i$$

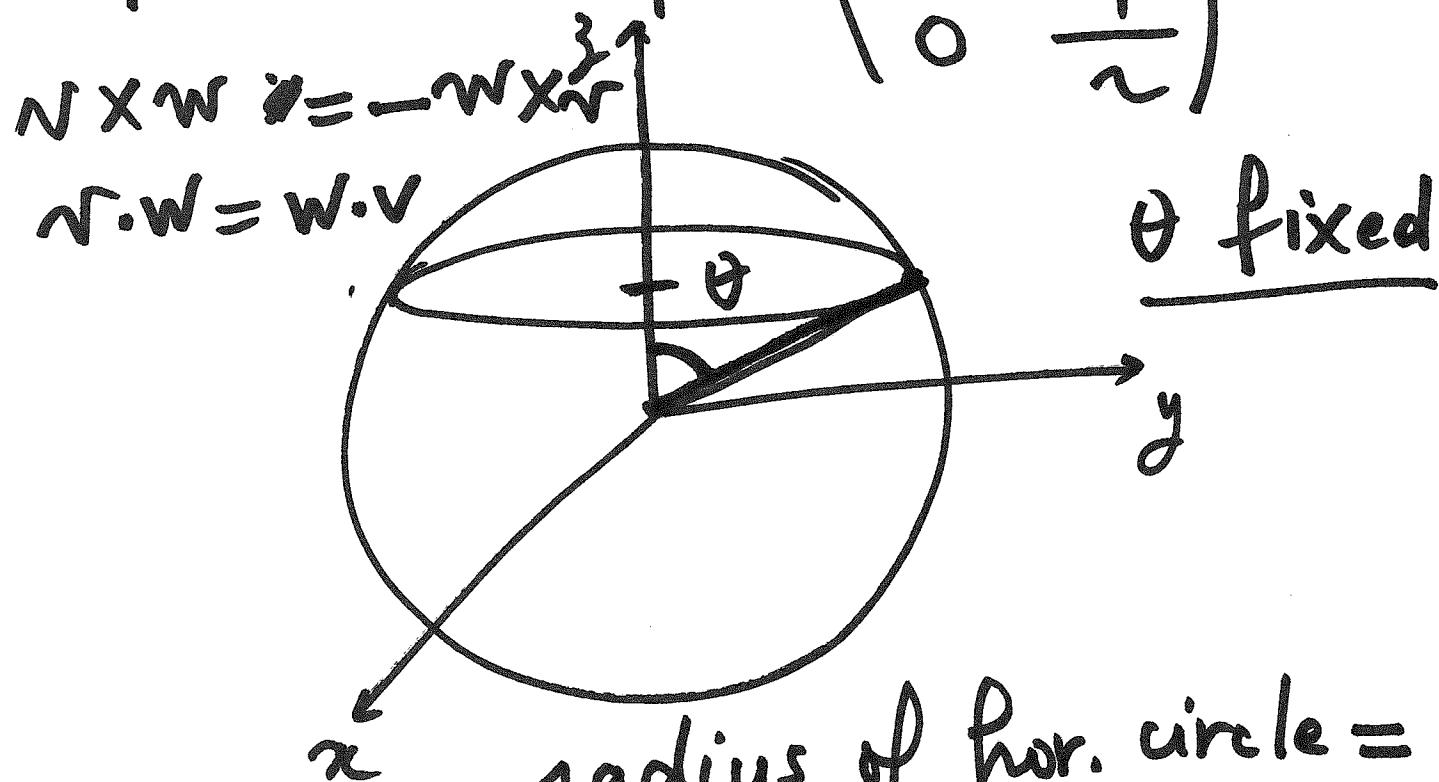
$$H \rightsquigarrow -H$$

$K \approx K$  independent  
of orientation

Ex: The sphere:

geographic param.:  $S_p = \begin{pmatrix} -\frac{1}{n} & 0 \\ 0 & -\frac{1}{n} \end{pmatrix}$

spherical:  $S_p = \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{n} \end{pmatrix}$



radius of hor. circle =  
 $r \sin \theta$ .

param:  $\alpha(\varphi) = (r \sin \theta \cos \varphi, r \sin \theta \sin \varphi, r \cos \theta)$

- (8) (a) Write the power series expansion (i.e., the Taylor series) of the function  $f(x) = \frac{1}{1-x^2}$  centered at  $c = -4$ .  
 (b) What is the radius of convergence of the power series above?

$$T \hookrightarrow \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \lambda_{11} \\ \lambda_{21} \end{pmatrix}$$

$$\textcircled{v_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \textcircled{v_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\lambda_{11} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \lambda_{21} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\underbrace{\qquad\qquad}_{v_1} \qquad \underbrace{\qquad\qquad}_{v_2}$$

$$v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ on the basis } v_1, v_2$$

$$= 1 \cdot v_1 + 0 v_2$$

- (8) (a) Write the power series expansion (i.e., the Taylor series) of the function  $f(x) = \frac{1}{1-x^2}$  centered at  $c = -4$ .  
 (b) What is the radius of convergence of the power series above?

$T_p M$

tangent space.

$$A_{QR} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

$$v \mapsto A_{QR}(v)$$

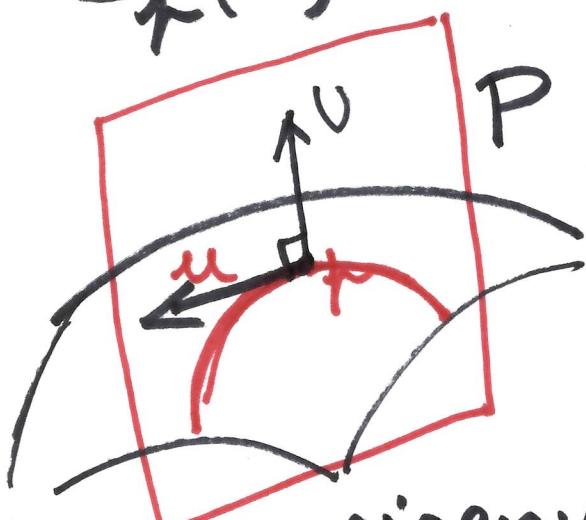
$$A_{QR}(\lambda v_1 + \mu v_2) = \lambda A_{QR}(v_1) + \mu A_{QR}(v_2)$$

Recall:  $M$  surface.  $p \in M$

$u$  = unit vector in  $T_p M$

the normal curvature of  $M$  in the direction of  $u$  is

$$S_p(u) \cdot u = -\nabla_u U \cdot u$$



$u_1$  &  $u_2$  eigenvectors for  $S_p$

$k_1$ ,  $k_2$  corresponding eigenvalues

$$S_p(u_1) \cdot u_1 = k_1 u_1 \cdot u_1 = k_1$$

↳ principal curvatures.

$C = P \cap M$ .  $\alpha$  par. by arc length

$$\alpha(0) = p \quad S_p(\alpha') \cdot \alpha' = -\frac{\partial U}{\partial s} \cdot \alpha'$$

$$\alpha' = T$$

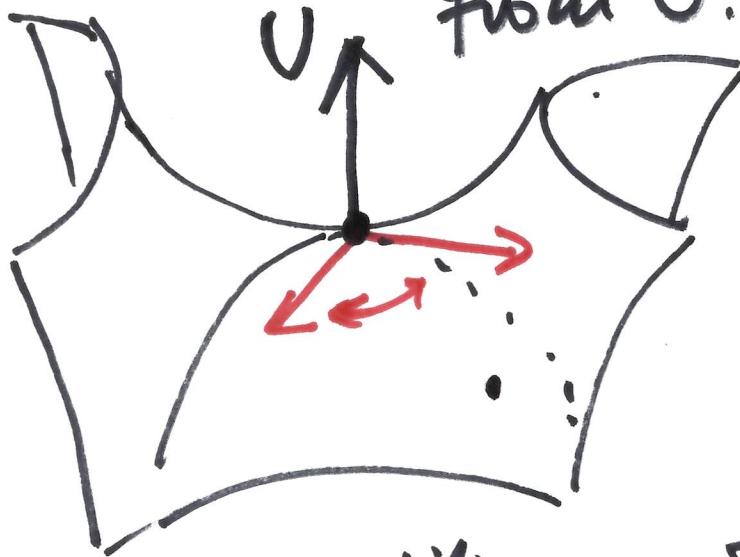
$$= \alpha'' \cdot U$$

$$\alpha'' = T' = KN \quad \stackrel{\downarrow}{=} KN \cdot U$$

$\mathbb{C}$  := normal section  $= K \cos \theta = \pm K$

$k(\alpha') > 0$  means  $M$  curves towards  $U$

$k(\alpha') < 0$   $M$  curves away from  $U$ .



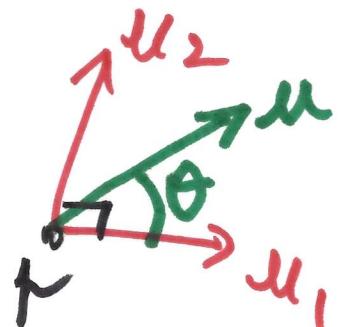
If  $t$  is umbilic:  $S_p = k \text{Id}$   
every direction is principal

If  $t$  is not umbilic  $k_1 \neq k_2$

$$M_1 \perp M_2$$

$k_1, k_2, M_1, M_2$  are intrinsic to  
the surface: do not depend on  
the choice of parametrization  
(given an orientation)

## Euler's formula (2.4.11) :



$u = \cos \theta u_1 + \sin \theta u_2$ , then

$$k(u) = k_1 \cos^2 \theta + k_2 \sin^2 \theta.$$

Proof:  $k(u) = S_p(u) \cdot u$

$$= S_p(\cos \theta u_1 + \sin \theta u_2) \cdot (\cos \theta u_1 + \sin \theta u_2)$$

$$= (\cos \theta S_p(u_1) + \sin \theta S_p(u_2)) \cdot (\quad \quad \quad)$$

$$= (k_1 \cos \theta u_1 + k_2 \sin \theta u_2) \cdot (\cos \theta u_1 + \sin \theta u_2)$$

$$= k_1 \cos \theta (u_1 \cdot u_1) + k_1 \cos \theta \sin \theta u_1 \cdot u_2$$

$$+ k_2 \sin \theta \cos \theta u_2 \cdot u_1 + k_2 \sin^2 \theta (u_2 \cdot u_2)$$

$$= k_1 \cos^2 \theta + k_2 \sin^2 \theta$$

Assume  $k_1 \geq k_2$

$$\begin{aligned}
 k(u) &= k_1(1 - \sin^2 \theta) + k_2 \sin^2 \theta \\
 &= k_1 + (k_2 - k_1) \sin^2 \theta \leq k_1 \\
 &= k_1 \cos^2 \theta + k_2 (1 - \cos^2 \theta) \\
 &= k_2 + (k_1 - k_2) \cos^2 \theta \geq k_2 \\
 k_2 &\leq k(u) \leq k_1
 \end{aligned}$$

Corollary:  $k_1$  is the maximum normal curvature,  $k_2$  is the minimum normal curvature.

recall:  $K = k_1 k_2$  is the Gaussian curvature.

$K > 0$ : all. normal sections bend the same way.

$K < 0$ : saddle point: some normals bend one way, some the other way

$K=0$  : one of the principal curvatures is 0, all other normal sections bend the same way.

Definitions: (1) An asymptotic direction at  $p$  is a direction with 0 normal curvature.  
(exists when  $K \leq 0$ )

(2) An asymptotic curve is a curve for which  $\alpha'(p)$  is asymptotic

(3) A principal curve or a line of curvature is a curve for which  $T$  is  $u_1$  or  $u_2$  everywhere

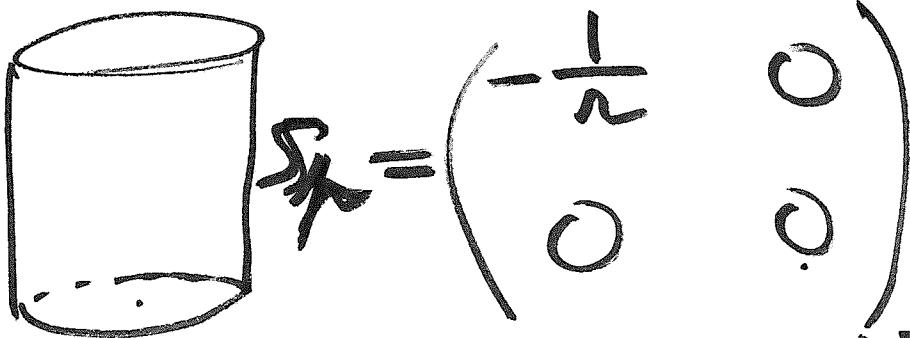
(4)  $M$  is called flat if  $K=0$  everywhere

(5)  $M$  is called minimal if  
 $H=0$  everywhere

$$(H = \frac{1}{2} (k_1 + k_2))$$

Examples: (1) Cylinders:

parametrization  $(r \cos u, r \sin u, v)$



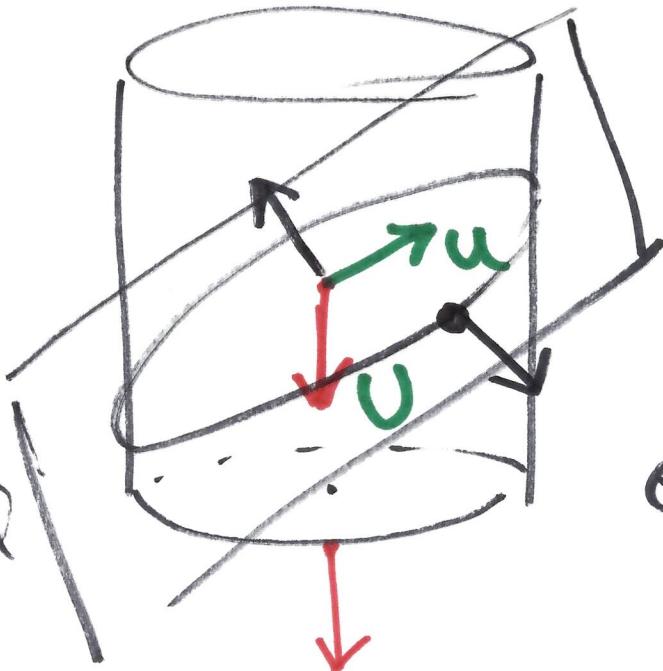
$$S_k = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & ? \end{pmatrix}$$

$K=0$  everywhere (at any point.  
homework).

Cylinders are flat

$$H = -\frac{1}{2r} \neq 0 \quad \text{Not minimal}$$

Normal sections:



Normal sections are ellipses NOT except principal curves.

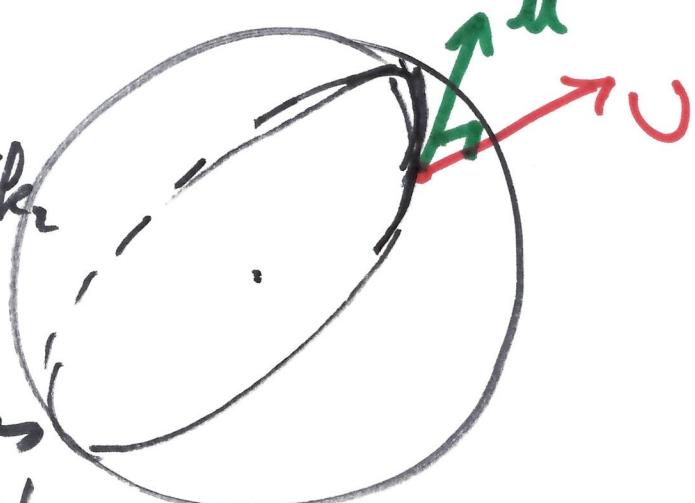
principal curves

- ↳ when (1) normal plane is horizontal circle
- (2) normal plane is vertical line.

(2) Spheres:  $\begin{pmatrix} \pm \frac{1}{r} & 0 \\ 0 & \pm \frac{1}{r} \end{pmatrix} = S_p$ .

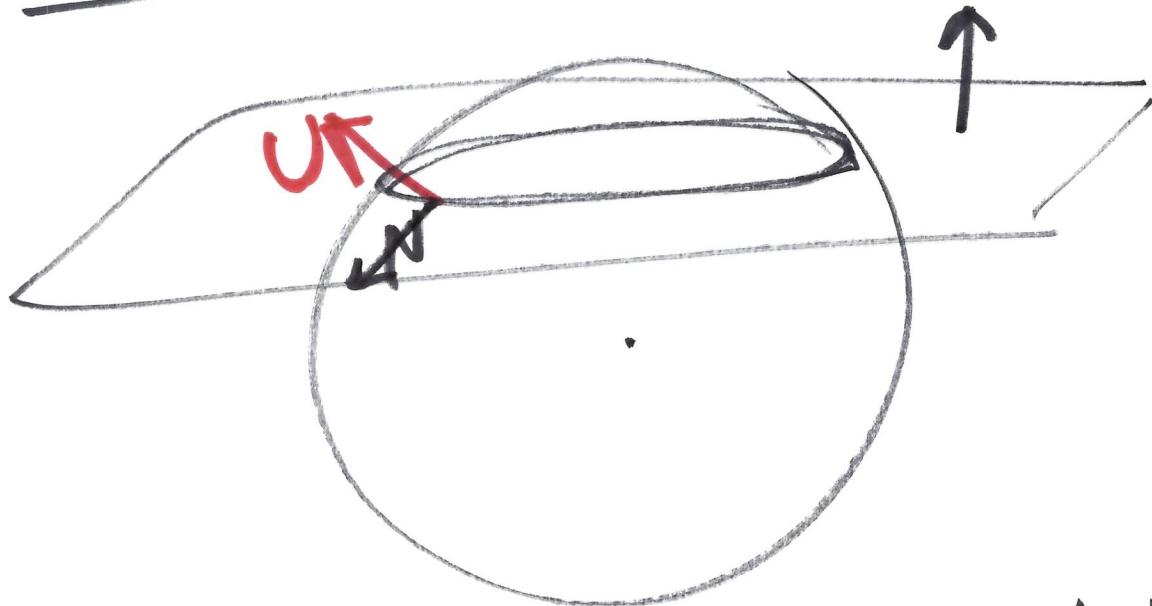
$$K = \frac{1}{r^2} > 0$$

$\forall u \quad k_1 \leq k(u) \leq k_2$   
 $k_1 = k_2 = k(u)$   
 all normal sections have the same shape



Normal Every point is umbilic  
 Normal sections are great circles.

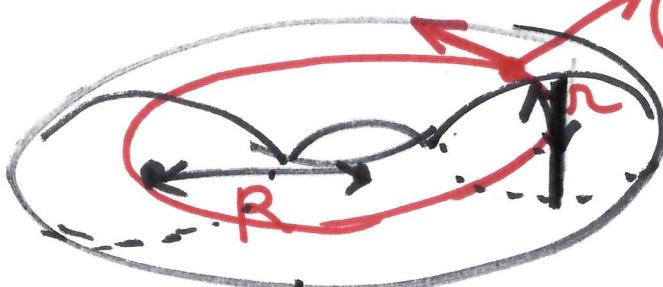
Principal curves:  $(\frac{Ex}{2}, \frac{4}{4}, \frac{4}{4})$



(3) Torus:  $S_p$  parametrization:

$$((R+r\cos u)\cos v, (R+r\cos u)\sin v, r\sin u)$$

$\rightarrow (r\cos u, r\sin u, 0)$



$$R > r > 0$$

$$S_p = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{\cos u}{R+r\cos u} \end{pmatrix}$$

$$K = \frac{\cos u}{r(R + r \cos u)} > 0$$

$K = 0$  when  $\cos u = 0$ .

Top circle & Bottom circle.  
where an asymptotic direction exists: it is  $\varphi_v / |\varphi_v|$

$$= \frac{1}{|\varphi_v|} ((R + r \cos u) (\sin v), (R + r \cos u) \cos v, 0)$$

$$= \frac{R + r \cos u}{|\varphi_v|} (-\sin v, \cos v, 0)$$

$\mu_2$  asymptotic

$K > 0$  when  $\cos u > 0$ .

outer half of torus:  $-\frac{\pi}{2} < u < \frac{\pi}{2}$

$K < 0$  when  $\cos u < 0$

saddle points  $\frac{\pi}{2} < u < \frac{3\pi}{2}$ .

inside half of torus

$$\boxed{T_p M} = \text{Span of } \boxed{\varphi_u}, \boxed{\varphi_v}$$

$$v = \lambda_u \varphi_u + \lambda_v \varphi_v$$

$$\left( \begin{array}{cc} -\frac{1}{2} & 0 \\ 0 & -\frac{\cos u}{2} \end{array} \right) \begin{pmatrix} \lambda_u \\ \lambda_v \end{pmatrix} = \begin{pmatrix} *_1 \\ *_2 \end{pmatrix}$$

$$= *_1 \varphi_u + *_2 \varphi_v$$

Spt.

$$\begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \varphi_u \\ \varphi_v \end{pmatrix}$$

at  $(u, v)$

~~$\varphi_u = v^2$~~

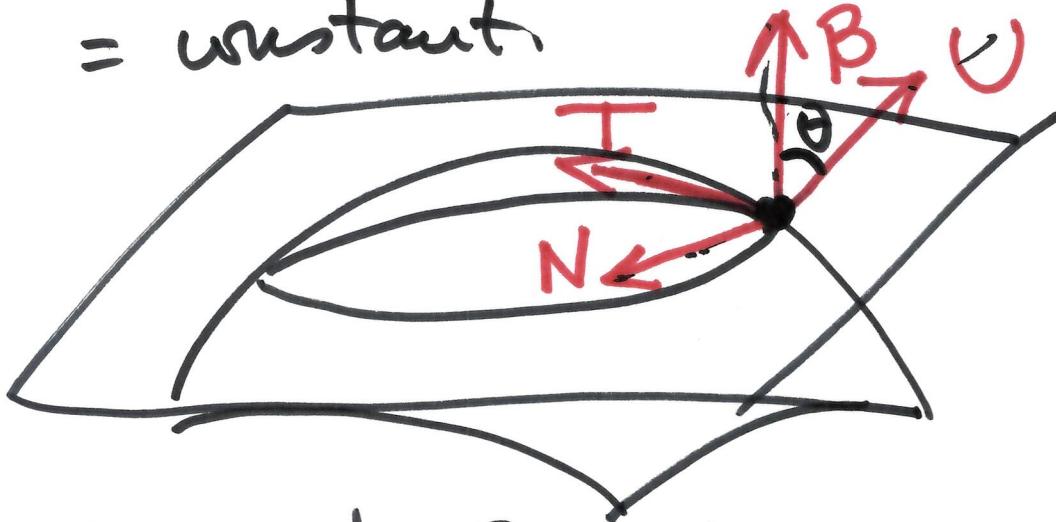
$$t \in C \subset M \cap P$$

$\theta = \text{angle between } V \text{ (normal to } M\text{)} \\ \text{ & normal to } P.$

$\theta = \text{constant}.$

$C \subset P \quad B = \text{binormal for } C \\ = \text{normal to } P.$

$\vartheta = \text{angle between } V \& B. \\ = \text{constant}$



$$T \perp, N, B, V$$

$\Rightarrow N, B, V \text{ are coplanar.}$

~~$$\Rightarrow H = \cos \theta B + \sin \theta N$$~~

$$V = \cos\left(\theta + \frac{\pi}{2}\right) N + \sin\left(\theta + \frac{\pi}{2}\right) B$$

$$V = -\sin \theta N + \cos \theta B.$$

Show:  $S_p(T) = k_i T \quad \forall p \in C?$

$$\begin{aligned} S_p(T) &= -\nabla_T U = -\frac{\partial U}{\partial s} \\ &= -\frac{\partial}{\partial s} \left( -\sin\theta N + \cos\theta B \right) \\ &= \sin\theta \frac{\partial N}{\partial s} - \cos\theta \frac{\partial B}{\partial s} \\ &= \sin\theta (-kT) + 0 \\ &= -k \sin\theta T \end{aligned}$$

$\Rightarrow C$  is a principal curve.

---

Theorem (3.5.2):

Suppose  $M$  is connected.  
(path connected)

If every point of  $M$  is umbilic,  
then  $M$  is contained in a  
sphere or a plane.

Proof:  $\forall p \in M, S_p = k_p \text{Id.}$   
where  $k_p = k_1 = k_2$ .

First goal: show  $k_F = \text{constant}$ .  
 (if MCP, then  $k_F = 0$  everywhere)  
 MCS, then  $k_F = \frac{\pm 1}{2}$ )

Choose a path  $\varphi(u, v)$  for M.

$$k_F = k(u, v).$$

Compute  $\frac{\partial k}{\partial u}$  &  $\frac{\partial k}{\partial v}$  and  
 show they are 0.

$$S_F(\varphi_u) = -\nabla_{\varphi_u} U = \boxed{-\frac{\partial U}{\partial u}}$$

$$= k \varphi_u$$

$$S_F(\varphi_v) = -\nabla_{\varphi_v} U = \boxed{-\frac{\partial U}{\partial v}}$$

$$= k \varphi_v.$$

$$\frac{\partial}{\partial v} \left( -\frac{\partial U}{\partial u} \right) = \frac{\partial}{\partial u} \left( -\frac{\partial U}{\partial v} \right)$$

~~$$\frac{\partial k}{\partial v} \varphi_u + k \varphi_{uv} = \frac{\partial k}{\partial u} \varphi_v + k \varphi_{uv}$$~~

$\varphi_u, \varphi_v$  linearly indep.

$$\Rightarrow \frac{\partial k}{\partial u} = \frac{\partial k}{\partial v} = 0$$

$\Rightarrow k$  is constant on patch.

Case 1:  $k = 0$

$$S_p(\varphi_u) = k \varphi_u = 0 = -\frac{\partial U}{\partial u}$$

$$S_p(\varphi_v) = k \varphi_v = 0 = -\frac{\partial U}{\partial v}$$

$\Rightarrow U$  is constant.

$$\frac{\partial}{\partial u}(U \cdot \varphi) = U \cdot \varphi_u = 0$$

$$\frac{\partial}{\partial v}(U \cdot \varphi) = U \cdot \varphi_v = 0$$

$\Rightarrow$   $U \cdot \varphi$  constant.

$$U = (u_1, \underline{u_2}, u_3) \text{ constant.}$$

$$\varphi = (x, y, z) = \varphi(u, v)$$

$$x = x(u, v) \dots$$

$$U \cdot \varphi = u_1 x + u_2 y + u_3 z = \text{const}_{\text{anti}}$$

$\Rightarrow$  image of  $\varphi$  is contained in a plane.

Case 2:  $k \neq 0$ .

$$Q := \varphi + \frac{1}{k} v$$

Show  $Q$  is constant:

$$\frac{\partial Q}{\partial u} = \varphi_u + \frac{1}{k} \frac{\partial v}{\partial u} = 0$$

because  $-\frac{\partial v}{\partial u} = k \varphi_u$

$$\frac{\partial Q}{\partial v} = \varphi_v + \frac{1}{k} \frac{\partial v}{\partial v} = 0$$

So  $Q$  is constant.

$$\varphi - Q = -\frac{1}{k} v.$$

$$|\varphi - Q| = \left| \frac{1}{k} v \right| \text{ constant.}$$

So  $\varphi(u, v) \in$  ~~center~~  
↑  $Q$  & radius  
 sphere.  $\left| \frac{1}{k} v \right|$

The above was for one patch.  
 If we have two patches, then

they intersect in a surface.

So, if one patch is contained in a plane, then the other is ~~not~~ contained in the same plane.

If one patch is contained in a sphere, the other is contained in the same sphere.

---

Competing curvatures:

$$\varphi(u, v)$$

$$S_f(\varphi_u) = a \varphi_u + b \varphi_v$$

$$S_f(\varphi_v) = c \varphi_u + d \varphi_v$$

matrix of  $S_f$  in  $\{\varphi_u, \varphi_v\}$ :

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \text{matrix of } S_f$$

$$K = ad - bc, H = \frac{1}{2}(a+d)$$

$$\boxed{S_f(\varphi_u) \times S_f(\varphi_v) =}$$

$$(a\varphi_u + b\varphi_v) \times (c\varphi_u + d\varphi_v)$$

$$= ad \varphi_u \times \varphi_v + bc \varphi_v \times \varphi_u$$

$$= (ad - bc) \varphi_u \times \varphi_v \quad \boxed{= K \varphi_u \times \varphi_v}$$

$$S_f(\varphi_u) \times \varphi_v \neq \cancel{\varphi_u \times S_f(\varphi_v)}$$

$$= (a\varphi_u + b\varphi_v) \times \varphi_v \neq (c\varphi_u + d\varphi_v) \times \varphi_u$$

$$= a \varphi_u \times \varphi_v \neq d \varphi_v \times \varphi_u$$

$$= (a+d) \varphi_u \times \varphi_v \quad \boxed{= 2H \varphi_u \times \varphi_v}$$

$$= S_f(\varphi_u) \times \varphi_v + \varphi_u \times S_f(\varphi_v)$$

$$(S_f(\varphi_u) \times S_f(\varphi_v)) \cdot (\varphi_u \times \varphi_v)$$

$$= K (\varphi_u \times \varphi_v) \cdot (\varphi_u \times \varphi_v)$$

$$= K |\varphi_u \times \varphi_v|^2.$$

Lagrange's formula (Ex. 3.1.5)

For vectors  $a, b, v, w$ ,

$$(v \times w) \cdot (a \times b) = (v \cdot a)(w \cdot b) - (v \cdot b)(w \cdot a)$$

$$\Rightarrow (\Sigma_L(\varphi_u) \times \Sigma_F(\varphi_v)) \cdot (\varphi_u \times \varphi_v)$$

$$= (\Sigma_L(\varphi_u) \cdot \varphi_u)(\Sigma_F(\varphi_v) \cdot \varphi_v) = l_n$$

$$- (\Sigma_F(\varphi_u) \cdot \varphi_v)(\Sigma_F(\varphi_v) \cdot \varphi_u) = m^2$$

---

$$(\varphi_u \times \varphi_v) \cdot (\varphi_u \times \varphi_v) =$$

$$(\varphi_u \cdot \varphi_u)(\varphi_v \cdot \varphi_v) - (\varphi_u \cdot \varphi_v)(\varphi_v \cdot \varphi_u)$$

Def.  $E := \varphi_u \cdot \varphi_u$

$$F := \varphi_u \cdot \varphi_v = \varphi_v \cdot \varphi_u$$

$$G := \varphi_v \cdot \varphi_v$$

the metric coefficients  
or the coeff. of first fundamental form.

$$l := \Sigma_F(\varphi_u) \cdot \varphi_u$$

$$m := \Sigma_F(\varphi_u) \cdot \varphi_v = \Sigma_F(\varphi_v) \cdot \varphi_u$$

$$\begin{matrix} \\ \\ \cup \\ \end{matrix} \cdot \varphi_{uv}$$

$$n := \Sigma_F(\varphi_v) \cdot \varphi_v$$

$$l_n - m^2 = K (E G - F^2)$$

$$\Rightarrow K = \frac{l_n - m^2}{E G - F^2}$$

Similarly:  $\partial H, (\varphi_u \times \varphi_v) \cdot (\varphi_u \times \varphi_v)$

$$= (\Sigma_F(\varphi_u) \times \varphi_v) \cdot (\varphi_u \times \varphi_v) +$$

$$(\varphi_u \times \Sigma_F(\varphi_v)) \cdot (\varphi_u \times \varphi_v)$$

Use Lagrange again:

$$(\Sigma_F(\varphi_u) \times \varphi_v) \cdot (\varphi_u \times \varphi_v) =$$

$$\begin{aligned}
&= (\alpha \varphi_u + \beta \varphi_v) \cdot \varphi_u \\
&= \alpha \varphi_u \cdot \varphi_u + \beta \varphi_v \cdot \varphi_u \\
&= \alpha E + \beta F \\
m &= \Sigma_h(\varphi_u) \cdot \varphi_v = \Sigma_h(\varphi_v) \cdot \varphi_u \\
&\quad (= U \cdot \varphi_{uv}) \\
&= (\alpha \varphi_u + \beta \varphi_v) \cdot \varphi_v = (c \varphi_u + d \varphi_v) \cdot \varphi_u \\
&= \alpha F + \beta G = c E + d F \\
n &= \Sigma_h(\varphi_v) \cdot \varphi_v = (c \varphi_u + d \varphi_v) \cdot \varphi_v \\
&= c F + d G \\
\begin{pmatrix} e & m \\ m & n \end{pmatrix} &= \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} E & F \\ F & G \end{pmatrix} \\
&= \begin{pmatrix} E & F \\ F & G \end{pmatrix} \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{S_F}
\end{aligned}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

$$a = \frac{1}{EG - F^2} (Ge - Fm)$$

$$d = \frac{1}{EG - F^2} (-Fm + En)$$

$$\varphi(u, v) = (u, v, 0)$$

$$\varphi_u = (1, 0, 0) \quad \varphi_v = (0, 1, 0)$$

$$\varphi_u \times \varphi_v = (0, 0, 1) = \pm U = v$$

$$E = \varphi_u \cdot \varphi_u = 1 \quad F = \varphi_u \cdot \varphi_v = 0$$

$$G = \varphi_v \cdot \varphi_v = 1$$

$$l = \Sigma_F (\varphi_u) \cdot \varphi_u = v \cdot \varphi_{uu} = 0$$

$$m = \Sigma_F (\varphi_u) \cdot \varphi_v = v \cdot \varphi_{uv} = 0$$

$$n = \Sigma_F (\varphi_v) \cdot \varphi_v = v \cdot \varphi_{vv} = 0$$

$$K = 0 = H = k_1 = k_2$$

$$(\Sigma_h(\varphi_u) \cdot \varphi_u)(\varphi_v \cdot \varphi_v) - (\Sigma_h(\varphi_u) \cdot \varphi_v) \\ (\varphi_v \cdot \varphi_u)$$

$$= \ell G - m F$$

$$(\varphi_u \times \Sigma_k(\varphi_v)) \cdot (\varphi_u \times \varphi_v) = \\ (\varphi_u \cdot \varphi_u)(\Sigma_k(\varphi_v) \cdot \varphi_v) - \\ (\varphi_u \cdot \varphi_v)(\Sigma_k(\varphi_v) \cdot \varphi_u)$$

$$= E_n - F_m$$

$$2H(EG - F^2) = \ell G - m F + E_n - m F$$

$$H = \frac{G\ell + E_n - 2F_m}{2(EG - F^2)}$$

Exercise 3.1.7:

$$k_i = H \pm \sqrt{H^2 - K}$$

$$\ell = \Sigma_k(\varphi_u) \cdot \varphi_u \in U \cdot \varphi_u \\ (2.3.5) \& 2.4.1$$

$$\Rightarrow \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} l & m \\ n & o \end{pmatrix}$$

So we get the matrix of the shape operator.

Remark: (Ex. 3.2.7)

The curves  $u = \text{constant}$  or  $v = \text{constant}$  (the coordinate curves) are principal curves ( $\Leftrightarrow F = m = 0$  (provided  $M$  has isolated umbilic points))

The principal curvatures are

then  $a = \frac{Gl}{EG} = \frac{l}{E} = k_1$

$$d = \frac{En}{EG} = \frac{n}{G} = k_2, \text{ note.}$$

Example: (1) Plane

(2) Sphere: spherical coord . . .

$$\varphi(u, v) = r(\cos u \sin v, \sin u \sin v, \cos v)$$

$$\varphi_u = r(-\sin u \sin v, \cos u \sin v, 0)$$

$$\varphi_v = r(\cos u \cos v, \sin u \cos v, -\sin v)$$

$$\varphi_{uu} = r(-\cos u \sin v, -\sin u \sin v, 0)$$

$$\varphi_{uv} = r(-\sin u \cos v, \cos u \cos v, 0)$$

$$\varphi_{vv} = r(\cos u \sin v, -\sin u \sin v, -\cos v)$$

$$U = (\cos u \sin v, \sin u \sin v, \cos v)$$

$$E = r^2 (\sin^2 u \sin^2 v + \cos^2 u \sin^2 v)$$
$$= r^2 \sin^2 v$$

$$F = 0 \quad G = \frac{r^2}{\sin^2 v}$$

$$l = U \cdot \varphi_{uu} = -r \sin^2 v$$

$$m = U \cdot \varphi_{uv} = 0 \quad \text{neglect}$$

$$n = U \cdot \varphi_{vv} = -r$$

$$K = \frac{En - m^2}{EG - F^2} = \frac{En}{EG} = \frac{r^2 \sin^2 v}{r^4 \sin^2 v}$$

$$= \frac{1}{r^2}$$

$$H = \frac{Gl + En - 2Fm}{2(EG - F^2)} = \frac{Gl + En}{2EG}$$

$$= \frac{r^2(-r \sin^2 v) + r^2 \sin^2 v (-r)}{2r^2 \sin^2 v \cdot r^2}$$

$$= -\frac{1}{r} = (\leftarrow k_1 = k_2)$$

$$k_i = H \pm \sqrt{H^2 - K}$$

$$= -\frac{1}{r} \pm \sqrt{\frac{1}{r^2} - \frac{1}{r^2}}$$

$$= -\frac{1}{r}$$

### (3) The paraboloid

$$\varphi(u, v) = (u, v, u^2 + v^2)$$

$$\varphi_u = (1, 0, 2u) \quad \varphi_v = (0, 1, 2v)$$

$$\varphi_{uu} = (0, 0, 2), \varphi_{uv} = (0, 0, 0)$$

$$\varphi_{vv} = (0, 0, 2)$$

$$\begin{vmatrix} i & j & k \\ 0 & 0 & 2u \\ 0 & 1 & 2v \end{vmatrix} = i(-2u) - j(2v) + k(1) = (-2u, -2v, 1)$$

$$U = \frac{1}{\sqrt{1+4u^2+4v^2}} (2u, -2v, 1)$$

$$E = 1+4u^2 \quad F = 4uv$$

$$G = 1+4v^2$$

$$\ell = U \cdot \varphi_{uu} = \frac{2}{\sqrt{1+4u^2+4v^2}}$$

$$m = U \cdot \varphi_{uv} = 0$$

$$n = U \cdot \varphi_{vv} = \frac{2}{\sqrt{1+4u^2+4v^2}}$$

$$K = \frac{4}{(1+4u^2+4v^2)^2}$$

$$H = \frac{1}{(1+4u^2+4v^2)^{3/2}} + \frac{1}{(1+4u^2+4v^2)^{1/2}}$$

$$k_1 = \frac{2}{(1+4u^2+4v^2)^{3/2}}$$

$$k_2 = \frac{2}{(1+4u^2+4v^2)^{1/2}}$$

proof: add and divide by 2  
to get H  
multiply to get K.

#### (4) Surfaces of revolution:

$$\alpha(tu) = (g(tu), h(u)) \subset xy \text{ plane}$$

rotate around x-axis:

$$\varphi(u, v) = (g(u), h(u)\cos v, h(u)\sin v)$$

$u = \text{constant}$ , circle  $\parallel yz$  plane  
this is a parallel.

$\nu$  = constant : copy of the original curve

this is a meridian

$$\varphi_u = (g', h' \cos \nu, h' \sin \nu)$$

$$\varphi_v = (0, -h \sin \nu, h \cos \nu)$$

$$\varphi_u \times \varphi_v = \begin{vmatrix} i & j & k \\ g' & h' \cos \nu & h' \sin \nu \\ 0 & -h \sin \nu & h \cos \nu \end{vmatrix}$$

$$= (hh', -g'h \cos \nu, -g'h \sin \nu)$$

$$U = \pm \frac{(hh', -g'h \cos \nu, -g'h \sin \nu)}{h \sqrt{g'^2 + h'^2}}$$

$$U = \pm \frac{(h', -g' \cos \nu, -g' \sin \nu)}{\sqrt{g'^2 + h'^2}}$$

$$\varphi_{uu} = (g'', h'' \cos \nu, h'' \sin \nu)$$

$$\varphi_{uv} = (0, -h' \sin \nu, h' \cos \nu)$$

$$\varphi_{vv} = (0, -h \cos v, -h \sin v)$$

$$E = g'^2 + h'^2 \quad F = 0.$$

$$G = h^2.$$

$$\ell = V \cdot \varphi_{uu} = \frac{h'g'' - h''g'}{\sqrt{g'^2 + h'^2}}$$

$$m = V \cdot \varphi_{uv} = \frac{0}{\cancel{\sqrt{g'^2 + h'^2}}}$$

$$n = V \cdot \varphi_{vv} = \frac{g'h}{\sqrt{g'^2 + h'^2}}$$

$$K = \frac{\ln - m^2}{EG - F^2} = \frac{\ln}{EG} = \frac{g'h(h'g'' - h''g')}{h^2(g'^2 + h'^2)^2}$$

$$K = \frac{g'(h'g'' - h''g')}{h(g'^2 + h'^2)^2}$$

$$H = \frac{\ell G + n E - 2 m F}{2(EG - F^2)} = \frac{\ell G + n E}{2(EG - F^2)}$$

$$H = \frac{R^2(h'g'' - h''g') + gh(g'^2 + R^2)}{2h^2(g'^2 + h'^2)^{3/2}}$$

$$k_1 = \frac{\ell}{E} = \frac{h'g'' - h''g'}{(g'^2 + h'^2)^{3/2}}$$

$$k_2 = \frac{n}{G} = \frac{g'h}{h^2(g'^2 + h'^2)^{1/2}} = \frac{g'}{....}$$

double check:  $K = k_1 k_2$

$$H = \frac{1}{2}(k_1 + k_2)$$

If we have arc length parameterization for  $\alpha$ :  $|\alpha'| = 1 = \sqrt{g'^2 + h'^2}$

$$\text{So } g'^2 + h'^2 = 1$$

$$\Rightarrow k_1 = h'g'' - h''g'$$

$$k_2 = \frac{g'}{h}$$

old exercise 1.4.6: plane curve

$$\alpha(t) = (x(t), y(t)) \mid x'y'' - x''y' \mid$$

curvature =

$$(x'^2 + y'^2)^{3/2}$$

$$K = \frac{g'}{h} (h'g'' - h''g')$$

$$H = \frac{h(h'g'' - h''g') + g'}{2h}$$

$$g'^2 + h'^2 = 1 \Rightarrow (g'^2 + h'^2)' = 0$$

$$2g'g'' + 2h'h'' = 0.$$

$$g'g'' = -h'h''$$

$$K = \frac{g'g''h' - h''g'^2}{g'g''h' - h''g'^2}$$

$$= -\frac{h'^2h''}{h^2} - h''g'^2$$

$$K = -\frac{h''}{h^2}$$

If  $\alpha$  is a graph:

$$\alpha(u) = (u, h(u))$$

$$\text{i.e., } g(u) = u \quad g' = 1, g'' = 0$$

$$E = 1 + h'^2 \quad F = 0 \quad G = h^2$$

$$l = \frac{\cancel{h''} - h''}{\sqrt{1+h'^2}}, m=0, n=\frac{h}{\sqrt{1+h'^2}}$$

$$K = \frac{\cancel{h''} - h''}{h(1+h'^2)}$$

$$H = \frac{h(\cancel{h''} - h'') + (1+h'^2)}{2h(1+h'^2)^{3/2}}$$

$$k_1 = \frac{\cancel{h''} - h''}{(1+h'^2)^{3/2}}, k_2 = \frac{1}{h(1+h'^2)^{1/2}}$$

$\begin{pmatrix} E & F \\ F & G \end{pmatrix}$  metric or first fundamental form

Second fundamental form:

$V, W$  vectors:

$$\begin{aligned} I_2(V, W) &:= S_t(V) \cdot W \\ &= V \cdot S_t(W) \end{aligned}$$

Gauss' theorema Egregium.

(remarkable theorem).

If two surfaces are locally isometric, then they have the same Gaussian curvature.

NOT true for mean curvature.

Example: plane and cylinder are locally isometric but have different mean curvatures.

Theorem (3.4.1). The Gaussian curvature depends only on the metric. If  $F=0$ , then

$$K = -\frac{1}{2\sqrt{EG}} \left( \frac{\partial}{\partial r} \left( \frac{E_u}{\sqrt{EG}} \right) + \frac{\partial}{\partial u} \left( \frac{G_u}{\sqrt{EG}} \right) \right)$$

Proof:  $\varphi, \varphi_u, \varphi_v, u$   
 $\varphi_{uu} = \Gamma_{uu}^u \varphi_u + \Gamma_{uv}^u \varphi_v + \Gamma_{vu}^u \varphi_v + \Gamma_{vv}^u \varphi_v$

$$\frac{\partial}{\partial v} (\varphi_u \cdot \varphi_u) = \varphi_{uv} \cdot \varphi_u + \varphi_u \cdot \varphi_{uv}$$

$$E_v = 2 \varphi_{uv} \cdot \varphi_u$$

$$\varphi_{uu} \cdot \varphi_v ? \quad F = \varphi_u \cdot \varphi_v$$

$$F_u = \frac{\partial}{\partial u} (\varphi_u \cdot \varphi_v) = \varphi_{uu} \cdot \varphi_v + \underbrace{\varphi_u \cdot \varphi_{uv}}$$

$$\varphi_{uu} \cdot \varphi_v = F_u - \frac{1}{2} E_v \quad \frac{1}{2} E_v$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{uu}^u \\ \Gamma_{uu}^v \end{pmatrix} = \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{uv}^u \\ \Gamma_{uv}^v \end{pmatrix} = \frac{1}{2} \begin{pmatrix} E_v \\ G_u \end{pmatrix}$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} \Gamma_{vv}^u \\ \Gamma_{vv}^v \end{pmatrix} = \begin{pmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{pmatrix}$$

$$\begin{pmatrix} \Gamma_{uu}^u \\ \Gamma_{uu}^v \end{pmatrix} = \frac{1}{EG-F^2} \begin{pmatrix} G-F & -F \\ -F & E \end{pmatrix} \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix}$$

$$\ell = \varphi_{uu} \cdot U = 2U \cdot U = 2$$

$$\varphi_{uu} = \Gamma_{uu}^u \varphi_u + \Gamma_{uu}^v \varphi_v + \ell U$$

$$\varphi_{uv} = \Gamma_{uv}^u \varphi_u + \Gamma_{uv}^v \varphi_v + m U$$

$$\varphi_{vv} = \Gamma_{vv}^u \varphi_u + \Gamma_{vv}^v \varphi_v + n U$$

$$\varphi_{uu} \cdot \varphi_u = \Gamma_{uu}^u E + \Gamma_{uu}^v F = \frac{1}{2} E_u$$

$$\varphi_{uu} \cdot \varphi_v = \Gamma_{uu}^u F + \Gamma_{uu}^v G = F_u - \frac{E_v}{2}$$

$$\varphi_{uv} \cdot \varphi_u = \Gamma_{uv}^u E + \Gamma_{uv}^v F = \frac{1}{2} E_v$$

$$\varphi_{uv} \cdot \varphi_v = \Gamma_{uv}^u F + \Gamma_{uv}^v G = \frac{1}{2} G_u$$

$$\varphi_{vv} \cdot \varphi_u = \Gamma_{vv}^u E + \Gamma_{vv}^v F = F_v - \frac{G_u}{2}$$

$$\varphi_{vv} \cdot \varphi_v = \Gamma_{vv}^u F + \Gamma_{vv}^v G = \frac{1}{2} G_v$$

$$\frac{\partial}{\partial u} (\varphi_u \cdot \varphi_u) = 2 \varphi_{uu} \cdot \varphi_u$$

$$E_u = 2 \varphi_{uu} \cdot \varphi_u$$

$$\varphi_{uuu} = \varphi_{uvu}$$

$$F(\Gamma_{uu}^u \varphi_u)_v + (\Gamma_{uu}^v \varphi_v)_v + \ell U_v =$$

$$(\Gamma_{uv}^u \varphi_u)_u + (\Gamma_{uv}^v \varphi_v)_u + m U_u$$


---

$$\Gamma_{uuu}^u \varphi_u + \Gamma_{uu}^u \varphi_{uv} + \Gamma_{uuu}^v \varphi_v$$

$$+ \Gamma_{uu}^v \varphi_{vv} + l_v U + l U_v =$$

$$= \Gamma_{uvu}^u \varphi_u + \Gamma_{uv}^u \varphi_{uu} + \Gamma_{uvu}^v \varphi_v$$

$$+ \Gamma_{uv}^v \varphi_{uv} + m_u U + m U_u$$


---

$$F = 0. \quad \Gamma_{uu}^u = \frac{1}{2} \frac{E_u}{E}$$

$$\Gamma_{uu}^v = -\frac{1}{2} \frac{E_v}{G}$$

$$\Gamma_{uv}^u = \frac{1}{2} \frac{E_v}{E} \quad \Gamma_{uv}^v = \frac{1}{2} \frac{G_u}{G}$$

$$\Gamma_{vv}^u = -\frac{1}{2} \frac{G_u}{E} \quad \Gamma_{vv}^v = \frac{1}{2} \frac{G_v}{G}$$

$$\begin{aligned}
 (\varphi_{uu})_v &= \boxed{\varphi_{uuv} = \varphi_{uvu}} = (\varphi_{uv})_u \\
 (\varphi_{vv})_u &= \boxed{\varphi_{vvu} = \varphi_{uvv}} = (\varphi_{uv})_v \\
 (U_u)_v &= \boxed{U_{uv} = U_{vu}} = (U_v)_u
 \end{aligned}$$

$$U_u = \nabla_{\varphi_u} U = -\Sigma_F(\varphi_u).$$

$$U_u = -a\varphi_u - b\varphi_v$$

recall:  $\begin{pmatrix} l & m \\ m & n \end{pmatrix} = \underbrace{\begin{pmatrix} E & F \\ F & G \end{pmatrix}}_{\text{metric}} \underbrace{\begin{pmatrix} a & c \\ b & d \end{pmatrix}}_{\Sigma_F}$

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \frac{1}{EG-F^2} \begin{pmatrix} G-F \\ F-E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}$$

$$U_u = \frac{-Gl+Fm}{EG-F^2} \varphi_u + \frac{Fl-Em}{EG-F^2} \varphi_v$$

$$U_v = \frac{-Gm+Fn}{EG-F^2} \varphi_u + \frac{Fm-En}{EG-F^2} \varphi_v$$

Acceleration formulas when  $F=0$

$$\varphi_{uu} = \frac{1}{2} \frac{E_u}{E} \varphi_u - \frac{1}{2} \frac{E_v}{G} \varphi_v + \ell u$$

$$\varphi_{uv} = \frac{1}{2} \frac{E_v}{E} \varphi_u + \frac{1}{2} \frac{G_u}{G} \varphi_v + m v$$

$$\varphi_{vv} = -\frac{1}{2} \frac{G_u}{E} \varphi_u + \frac{1}{2} \frac{G_v}{G} \varphi_v + n v$$

$$U_u = -\frac{1}{E} \varphi_u - \frac{m}{G} \varphi_v$$

$$U_v = -\frac{m}{E} \varphi_u - \frac{n}{G} \varphi_v$$

$$\begin{aligned} & \left( \frac{1}{2} \frac{E_u}{E} \right)_v \underline{\varphi_u} + \frac{1}{2} \frac{E_u}{E} \left( \frac{1}{2} \frac{E_v}{E} \underline{\varphi_u} + \frac{1}{2} \frac{G_u}{G} \underline{\varphi_v} \right. \\ & \quad \left. + m v \right) \\ & + \left( -\frac{1}{2} \frac{E_v}{G} \right)_v \underline{\varphi_v} \\ & + \left( -\frac{1}{2} \frac{E_v}{G} \right) \left( -\frac{1}{2} \frac{G_u}{E} \underline{\varphi_u} + \frac{1}{2} \frac{G_v}{G} \underline{\varphi_v} + n v \right) \\ & + \ell_v \underline{u} + \ell \left( -\frac{m}{E} \underline{\varphi_u} - \frac{n}{G} \underline{\varphi_v} \right) \end{aligned}$$

$$= \lim_{E \rightarrow G} \dots$$



$$\begin{array}{c}
 ( \\
 + \\
 + \\
 = \\
 + \\
 +
 \end{array}
 \begin{array}{c}
 A_1 \\
 B_1 \\
 C_1 \\
 A_2 \\
 B_2 \\
 C_2
 \end{array}
 \begin{array}{c}
 ) \varphi_u \\
 ) \varphi_v \\
 ) v \\
 ) \varphi_u \\
 ) \varphi_v \\
 ) v
 \end{array}$$

$$\Rightarrow A_1 = A_2, \quad B_1 = B_2, \quad C_1 = C_2$$

$\Rightarrow$  3 identities between  $E, F, G$   
 $\ell, m, n$

each equation gives 3 identities.

$\Rightarrow$  total 9 identities.

Only 3 are independent

$\Rightarrow$   $\boxed{\ell, m, n}$  as functions  
 of  $\boxed{E, F, G}$