1. Review

1.1. **The Geometry of Curves.** A parametric curve in $\mathbb{R}^3$ is a map

$$\begin{aligned} \alpha : \ \mathbb{R} &\longrightarrow \mathbb{R}^3 \\ t &\longmapsto \alpha(t) := (x(t), y(t), z(t)). \end{aligned}$$

We say that α is differentiable if x, y, z are differentiable. We say that it is C^1 if, in addition, the derivatives are continuous. We say α is C^n if the first n derivatives exist and are continuous. We say that α is C^∞ or smooth if derivatives to any order exist.

The length of a parametric curve from t_1 to t_2 is the integral

$$\int_{t_1}^{t_2} |\alpha'(t)| dt$$

where $|\alpha'(t)| := \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}$ is the length of the *velocity vector* $\alpha'(t) := (x'(t), y'(t), z'(t))$.

The length of the vector $v := (a_1, b_1, c_1)$ is $|v| := \sqrt{a_1^2 + b_1^2 + c_1^2}$ which is also $\sqrt{v \cdot v}$ where $\cdot$ is the scalar or dot product. We have

$$v \cdot w := a_1 a_2 + b_1 b_2 + c_1 c_2.$$

Also,

$$v \cdot w = |v|\,|w| \cos(\theta)$$

where θ is the angle between v and w. The above leads to the **Schwarz inequality**:

$$|v \cdot w| \leq |v|\,|w|.$$

Theorem 1.1. *A line is the shortest curve between two points.*

1.2. **Arclength parametrization.** We say that a curve is parametrized by arclength s if the parameter s is the length of the curve from a certain point. Let $s(t) := \int_a^t |\alpha'(t)|\, dt$ be the length of the curve from the point $t = a$.

A reparametrization of the curve is a composition $\beta(t) = \alpha(u(t))$ where u is a function of a real variable. We say a curve is **regular** if $\alpha'(t)$ is never 0.

To reparametrize a curve by arclength, we must have

$$s(t) = t = \int_a^t |\beta'(t)|\, dt.$$

Taking derivatives of both sides, we obtain $|\beta'(t)| = 1$ by the fundamental theorem of Calculus. Conversely, if $|\beta'(t)| = 1$, then $s(t) = t$ and the curve is parametrized by arclength.

So to parametrize a curve by arclength means finding a parametrization such that the velocity vector always has length 1. We have

Theorem 1.2. *Every regular curve can be reparametrized by arclength.*

1.3. **Frenet frames.** A Frenet frame can be thought of as the path along which you would want three fingers to grab a curve so that it has no way of escaping! It is a coordinate system that moves on the curve. The first vector is the unit tangent vector. So it is most convenient to assume that the curve is parametrized by arclength so that the velocity vector automatically has length 1. The other two vectors should be perpendicular to the tangent vector but how should we choose them? They should be determined by the curve. One important property of the curve is how it turns. So the second vector will be a unit vector that tells us how the tangent vector changes. We define

$$N(t) := \frac{T'(t)}{|T'(t)|},$$

Assuming that the **curvature**

$$\kappa(s) := |T'(s)|$$

is not zero. The curvature tells us how fast the tangent vector turns. Note that by the product formula, because the length of T is constant, N is normal to T everywhere.

Definition 1.3. *The osculating plane at the point of parameter s is the plane spanned by the vectors $T(s)$ and $N(s)$.*

The third vector is the unique vector that completes the system T, N into a coordinate system. This vector is

$$B := T \times N$$

the cross-product of T and N. Recall that the cross-product of two vectors $v = (a_1, b_1, c_1)$ and $w = (a_2, b_2, c_2)$ can be defined as the "determinant"

$$v \times w = \begin{vmatrix} i & j & k \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

where i, j, k are the unit vectors $(1,0,0), (0,1,0), (0,0,1)$ respectively. The length of the cross-product is $|T||N|\sin(\theta)$. So B has length 1. The cross-product is bilinear, non associative:

$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C \neq (A \times B) \times C,$$

anti-commutative. If both v and w have length 1, their cross-product is the unit vector perpendicular to both of them, oriented with the thumb-screw rule.

The set (T, N, B) is the Frenet frame along β. We say that (T, N, B) is an **orthonormal** basis of $\mathbb{R}^3$. The variation of T, N and B will tell us how β twists and turns through space. Their variations are determined by their derivatives T', N' and B'.

We already know $T' = \kappa N$ by the definition of N. The **torsion** of the curve is:

$$\tau := N' \cdot B = -B' \cdot N.$$

We have the Frenet relations for a curve of unit speed:

$$\begin{array}{rcrcrcr} T' & = & & & \kappa N & & \\ N' & = & -\kappa T & & & & +\tau B \\ B' & = & & & -\tau N & & \end{array}$$

So the torsion is the component of the derivative of the normal vector N on the vector B. It tells us how the curve twists out of its osculating plane or how twisted the curve is! Torsion is the noun associated to the French verb "tordre" which means to twist.

Recall that $\kappa = |T'|$ is always ≥ 0. The curvature of a circle is the inverse of its radius.

Theorem 1.4. *Let β be a unit speed curve. Then*

(1) *$\kappa = 0$ if and only if β is a line.*
(2) *for $\kappa > 0$, $\tau = 0$ if and only if β is a plane curve.*

Theorem 1.5. *A curve β is part of a circle if and only if it is a plane curve ($\tau = 0$) and $\kappa > 0$ is constant.*

Definition 1.6. *For a general plane curve, the curve defined by*

$$\gamma(s) = \beta(s) + \frac{1}{\kappa(s)} N(s)$$

is the evolute of β: the locus of centers of its osculating circles.

1.4. **Non unit speed curves.** As we saw, it is not always easy to reparametrize a curve by arclength so we need to modify our Frenet formulas so that we can also compute with arbitrary parametrizations.

For this we have to recall

$$\beta(s) = \alpha(t).$$

The Frenet frame is defined in terms of a unit speed parametrization. So we theoretically switch to unit speed and then use the Chain rule to compute with an arbitrary parametrization. $T(s) = T(t)$ is always the unit tangent vector. So

$$T(t) = \frac{\alpha'(t)}{|\alpha'(t)|} = \beta'(s).$$

Recall that the curvature is

$$\kappa(s) := \left|\frac{dT}{ds}\right|.$$

If we don't have an explicit arclength parametrization, we can take derivatives with respect to t. Recall the formulas

$$\begin{aligned}
\tfrac{dT}{ds} &= && \kappa N && \\
\tfrac{dN}{ds} &= -\kappa T && && +\tau B \\
\tfrac{dB}{ds} &= && -\tau N. &&
\end{aligned}$$

Using the chain rule, we obtain

$$\begin{aligned}
\tfrac{dT}{dt} = \tfrac{dT}{ds}\tfrac{ds}{dt} &= && \kappa \tfrac{ds}{dt} N && \\
\tfrac{dN}{dt} = \tfrac{dN}{ds}\tfrac{ds}{dt} &= -\kappa \tfrac{ds}{dt} T && && +\tau \tfrac{ds}{dt} B \\
\tfrac{dB}{dt} = \tfrac{dB}{ds}\tfrac{ds}{dt} &= && -\tau \tfrac{ds}{dt} N. &&
\end{aligned}$$

First note that

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{1}{\nu(t)}\left|\frac{dT}{dt}\right|.$$

Computing $\alpha'(t), \alpha''(t)$ and $\alpha''(t)$ we obtain:

$$\kappa = \frac{|\alpha' \times \alpha''|}{\nu^3}, \qquad B = \frac{\alpha' \times \alpha''}{|\alpha' \times \alpha''|}$$

and we also obtain N as

$$N = B \times T$$

and τ as

$$\tau = \frac{N' \cdot B}{\nu}$$

or

$$\tau = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{\kappa^2 \nu^6} = \frac{(\alpha' \times \alpha'') \cdot \alpha'''}{|\alpha' \times \alpha''|^2}$$

since $\kappa\nu^3 = |\alpha' \times \alpha''|$.

Definition 1.7. *The* involute *of a plane curve* $\alpha(t)$ *is given by*

$$\mathcal{I}(t) = \alpha(t) - s(t)\frac{\alpha'(t)}{|\alpha'(t)|} = \alpha(t) - s(t)T(t)$$

and that the evolute *of a plane curve is given by*

$$\mathcal{E}(t) = \alpha(t) + \frac{N(t)}{\kappa(t)}.$$

These two operations are "inverses" to each other.

1.5. **The Fundamental theorem (1.5.17).**

Theorem 1.8. *Given smooth functions* $\kappa(s) > 0$ *and* $\tau(s)$ *on an interval* $I \subset \mathbb{R}$, *there exists a regular arclength parametrized curve* $\alpha : I \to \mathbb{R}$ *such that* κ *is its curvature and* τ *its torsion. Such a curve is unique up to a rigid motion (i.e. a combination of a rotation and a translation) of* $\mathbb{R}^3$.

1.6. **Green's theorem and the isoperimetric inequality.** A smooth curve $\alpha : [a,b] \to \mathbb{R}^2$ is closed if $\alpha^{(n)}(a) = \alpha^{(n)}(b)$ for all $n \geq 0$. Closed curves often represents periodic orbits of physical systems. A closed curve is simple if it does not cross itself, in other words, α is one-to-one on $[a,b)$.

Theorem 1.9. *(The isoperimetric inequality) Let* C *be a simple closed curve of length* L, *bounding a region of area* A. *Then*

$$L^2 \geq 4\pi A$$

with equality exactly when C *is a circle.*

For the proof we use the formula

$$A = \int_a^b x(t)y'(t)dt = -\int_b^a y(t)x'(t)dt.$$

If the curve can be divided into the union of the graphs of two functions f and g, then

$$A = \int_{x_1}^{x_2} g(x)dx - \int_{x_1}^{x_2} f(x)dx = -\int_{t_2}^{t_1} y(t)x'(t)dt - \int_{t_1}^{b} y(t)x'(t)dt - \int_{a}^{t_2} y(t)x'(t)dt = \int_a^b y(t)x'(t)dt.$$

Integration by parts yields the second formula:

$$\int_a^b y(t)x'(t)dt = [-y(t)x(t)]_a^b + \int_a^b x(t)y'(t)dt.$$

Or one can use

Theorem 1.10. *(Green's theorem) Let P and Q be smooth functions on a simply connected region (i.e., a region without holes) R of the plane with boundary a simple closed curve C. Then*

$$\iint_R \left(\frac{\partial P}{\partial y} + \frac{\partial Q}{\partial x} \right) dxdy = \int_C (Pdx - Qdy).$$

1.7. **Surfaces in $\mathbb{R}^3$.** Surfaces in $\mathbb{R}^3$ are unions of images of C^∞ (partials with respect to the two variables to any order exist) maps

$$\varphi : D \longrightarrow \mathbb{R}^3$$

where D is an open (meaning D does not contain its boundary) domain in $\mathbb{R}^2$, usually an open rectangle $]a,b[\times]c,d[$ or an open disc $|(x,y) - P| < r$. If u and v are the coordinates on D, we write $\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$ and define

$$\varphi_u := \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u} \right), \quad \varphi_v := \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v} \right).$$

A map φ is called *regular* if the derivatives φ_u, φ_v are linearly independent everywhere on D, i.e., the cross-product $\varphi_u \times \varphi_v$ is everywhere nonzero. The vector $\varphi_u \times \varphi_v$ is a normal to the surface and we can obtain a unit normal by dividing by its length.

We can usually make D smaller so that the map φ is one-to-one and regular. Such a map φ is then called a *coordinate chart or patch* for the surface. For such a map φ there exists an inverse (defined on the image of φ). If we have two coordinate charts φ and ψ we can form the composition

$$\psi^{-1} \circ \varphi : D \longrightarrow \mathbb{R}^2.$$

A surface is called *smooth* if for all coordinate charts φ, ψ, the composition $\psi^{-1} \circ \varphi$ is smooth. The composition $\psi^{-1} \circ \varphi$ is a *change of coordinates.*

A set of coordinate charts covering S is usually called an *atlas.*

1.8. **Graphs (or Monge patches).** $\varphi(u,v) = (u, v, f(u,v))$ for a function of two variables f. Always a coordinate patch if f is smooth.

1.9. **The implicit function theorem.** When is an implicit equation $f(x,y,z) = c$ a surface? If we could solve for z, we would be able to write $z = g(x,y)$. Then on the surface,

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$$

and

$$dz = \frac{-\left(\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy \right)}{\frac{\partial f}{\partial z}}.$$

Theorem 1.11. *(The implicit function theorem) Near any point* (a,b,c) *satisfying* $f(x,y,z) = c$ *and* $\frac{\partial f}{\partial z} \neq 0$, z *can be written as a smooth function of* x *and* y *whose partials are*

$$\frac{\partial z}{\partial x} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}, \quad \frac{\partial z}{\partial y} = \frac{-\frac{\partial f}{\partial y}}{\frac{\partial f}{\partial z}}.$$

c is called a regular value of f, if no point satisfies $f(x,y,z) = c$, $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$.

1.10. **Spheres.** $x^2 + y^2 + z^2 = 1$. Upper, lower, left, right hemispheres... $z = \sqrt{x^2 + y^2}$ etc.

Spherical coordinates: $\omega(\theta, \phi) = R(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$.

Geographical coordinates: $u = \theta$, $v = \pi/2 - \phi$.

Compute the derivatives and the unit normal. What points on the sphere do we miss? Are these coordinate patches? How should we define the domains D?

1.11. **Surfaces of revolution.** Regular curve $\alpha(u) = (g(u), h(u), 0)$. Rotate about the x-axis:

$$\varphi(u,v) = (g(u), h(u)\cos v, h(u)\sin v).$$

How should we define D?

1.12. **Ruled surfaces.** A surface is ruled if it has a parametrization

$$w(u,v) = \alpha(u) + v\beta(u)$$

where α and β are two curves. This may not be a coordinate patch, we might have to remove some points to get coordinate patches. Each time we fix u, we get a line in S.

Examples:

Cones: $w = P + v\beta$ where P is fixed (all the lines go through one point).

Saddles: $z = xy$ (doubly ruled).

Hyperboloid of one sheet: $x^2 + y^2 - z^2 = 1$ (doubly ruled, see exercises in the book).

1.13. **The geometry of surfaces.** Curves have tangent lines, surfaces have tangent planes. There are 3 equivalent ways of looking at the tangent spaces of a surface:

(1) The tangent plane to S at the point (a,b,c) is the union of all the tangent lines to curves in S through (a,b,c).
(2) The tangent plane is the plane through (a,b,c) spanned by the vectors φ_u, φ_v in a coordinate chart φ near (a,b,c).
(3) The tangent plane is the plane through (a,b,c) with normal vector $\varphi_u \times \varphi_v$.

The first one is not so easy to compute with useful for geometric arguments and it is nice because it shows that the tangent plane is independent of the choice of chart.

A curve defined on an interval $I = [a, b]$ lies on a surface S if the map $\alpha : I \to \mathbb{R}^3$ factors through S. In other words, $\alpha : I \to S \subset \mathbb{R}^3$. The datum of a curve in S is equivalent to the data of two smooth maps $u(t), v(t) : I \to \mathbb{R}$ such that $\alpha(t) = \varphi(u(t), v(t))$. (See Lemma 2.1.3)

Recall that the equation of a plane can be written using the dot product:

$$N \cdot (Q - P) = 0$$

where N is a normal vector to the plane, P is a fixed point on the plane and Q is a variable point on the plane.

We define the unit normal vector

$$U := \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|}.$$

Definition 1.12. *A surface is orientable if it has a consistent normal vector defined everywhere on it. Single coordinate patches are always orientable. Problems might occur when we have to use more than one patch: we need to be able to glue the normals from one patch to the other.*

Essentially, the surface is orientable if, when we are moving on any loop (simple closed curve), then the normal at the starting point is equal to the normal at the ending point.

Example:: The Möbius strip is not orientable. First patch

$$\varphi(u_1, v_1) = \left(\left(2 - v_1 \sin\left(\frac{u_1}{2}\right)\right)\sin u_1, \left(2 - v_1 \sin\left(\frac{u_1}{2}\right)\right)\cos u_1, v_1 \cos\left(\frac{u_1}{2}\right)\right)$$

where $u_1 \in]0, 2\pi[, v_1 \in]-1, 1[$. Second patch

$$\psi(u_2, v_2) = \left(\left(2 - v_2 \sin\left(\frac{\pi}{4} + \frac{u_2}{2}\right)\right)\sin u_2, -\left(2 - v_2 \sin\left(\frac{\pi}{4} + \frac{u_2}{2}\right)\right)\cos u_2, v_2 \cos\left(\frac{\pi}{4} + \frac{u_2}{2}\right)\right)$$

where $u_2 \in]0, 2\pi[, v_2 \in]-1, 1[$.

From now on we assume that our surfaces are orientable.

Definition 1.13. *The shape operator S for a surface M is defined as*

$$S_p(V) = -\nabla_V U$$

at a point $p \in M$ where $\nabla_V U$ is the directional derivative of U in the direction of V. In other words, if $U = (f, g, h)$ and $V = (a, b, c)$, then

$$\nabla_V U = (\nabla_V f, \nabla_V g, \nabla_V h) = \left(a\frac{\partial f}{\partial x} + b\frac{\partial f}{\partial y} + c\frac{\partial f}{\partial z}, a\frac{\partial g}{\partial x} + b\frac{\partial g}{\partial y} + c\frac{\partial g}{\partial z}, a\frac{\partial h}{\partial x} + b\frac{\partial h}{\partial y} + c\frac{\partial h}{\partial z}\right).$$

Theorem 1.14. *(Lemma 2.2.10) S_p is a linear transformation from T_pM to itself.*

Note that if M is contained in a plane, then U is constant and all its derivatives are 0. We say that S is path connected, if for any two points $p, q \in S$, there exists a curve $\alpha : [0,1] \to S$ such that $\alpha(0) = p$ and $\alpha(1) = q$. Conversely

Theorem 1.15. *Assume M is path connected. If $S_p = 0$ at every point of M, then M is contained in a plane.*

Example: The sphere of radius R: $\varphi(u,v) = R(\cos u \cos v, \sin u \cos v, \sin v)$. $S_p(V) = -\frac{V}{R}$ for all V.

1.14. **The linear algebra of surfaces.** The computation of the shape operator doesn't always give us geometric information about the surface, i.e., its shape (see the example of the saddle, homework 2.2.16). However, there are other things we can do with the shape operator that do give us information about the shape of the surface. Review a few notions from linear algebra:

Matrix of a linear transformation on a basis.

Eigenvalues and eigenvectors. Matrix on a basis of eigenvectors and diagonalization.

Determinant and Trace in terms of eigenvalues.

Definition 1.16. *We say that a linear operator T is symmetric if, for any vectors v and w, $T(v) \cdot w = v \cdot T(w)$.*

Definition 1.17. *(reminder) A basis is orthonormal if it consists of orthogonal (perpendicular) unit vectors.*

Exercise 2.3.4: With respect to an orthonormal basis, the matrix of a symmetric operator is symmetric.

Theorem 1.18. *(2.3.5) We have*

$$S(\varphi_u) \cdot \varphi_u = \varphi_{uu} \cdot U, \quad S(\varphi_u) \cdot \varphi_v = \varphi_{uv} \cdot U = S(\varphi_v) \cdot \varphi_u, \quad S(\varphi_v) \cdot \varphi_v = \varphi_{vv} \cdot U.$$

In particular, the shape operator is symmetric.

The shape operator S_p contains information about the acceleration/curvature of curves in M through p.

Lemma 1.19. *(2.4.1) For any curve $\alpha(t)$ in M,*

$$\alpha'' \cdot U = S(\alpha') \cdot \alpha'.$$

The scalar product $\alpha'' \cdot U$ is the part of the acceleration due to the bending of M. The above formula shows that it only depends on the tangent vector to the curve. If the curve has unit speed, it only depends on the tangent direction to the curve.
If α has unit speed and α' is an eigenvector of S_p, then

$$S_p(\alpha') = \lambda\alpha'$$

and

$$\lambda = \lambda\alpha' \cdot \alpha' = S_p(\alpha') \cdot \alpha' = \alpha'' \cdot U.$$

Definition 1.20. *Given a unit vector $\mathbf{u}$ in the tangent space T_pM, the (normal) curvature of M in the direction of $\mathbf{u}$ is*

$$k(\mathbf{u}) \coloneqq S_p(\mathbf{u}) \cdot \mathbf{u}.$$

This is the curvature of the curve $P \cap M$ where P is the plane through p and parallel to $U(p)$ and $\mathbf{u}$ (see below and 2.4.3 in the book).
The (real) eigenvalues k_1, k_2 of S_p are the principal curvatures of M at p. The corresponding unit (orthogonal) eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$ are the principal directions.
The determinant $K \coloneqq k_1k_2$ is the Gaussian curvature at p.
Half the trace $H \coloneqq \frac{1}{2}(k_1 + k_2)$ is the mean curvature at p.
A point where $k_1 = k_2$ is umbilic: $S_p = k_1 id$ and every direction is principal.

Note that S, k_1, k_2, H change sign under a change of orientation but not K.
The normal curvature is the normal component of acceleration: by the Frenet formulas, for a unit speed curve

$$k(\alpha') = S_p(\alpha') \cdot \alpha' = \alpha'' \cdot U = \kappa N \cdot U = \kappa \cos\theta$$

where θ is the angle between the surface normal and the curve normal (draw a picture).

Proposition 1.21. *(2.4.3) Let $\mathbf{u}$ be a unit vector and P the plane through p and parallel to $U(p)$ and $\mathbf{u}$. Let σ be the unit speed curve formed by $P \cap M$ with $\sigma(0) = p$. Then*

$$k(\mathbf{u}) = \pm\kappa_\sigma(0).$$