Ruled surfaces:

Definition: A surface is ruled if it has a parametrization

$$\varphi(u,v) = \beta(u) + v\,\delta(u)$$

where β, δ are curves.

Note: for fixed $u = u_0$:

$$\varphi(u_0, v) = \beta(u_0) + v(\delta(u_0))$$

is a parametrization of a line on the surface. This lines goes through $\beta(u_0)$ and is parallel to $\delta(u_0)$.

Terminology: The (family) union of all the lines as above is called a ruling of the surface. The curve $\beta(u)$ is called a directrix.

We say the surface is doubly ruled if it has two rulings, meaning two parametrizations

$$\varphi_1(u,v) = \beta_1(u) + v\,\delta_1(u)$$

$$\varphi_2(u,v) = \beta_2(u) + v\,\delta_2(u).$$

<u>Remark</u>: the parametrizations are not always coordinate charts (i.e., there can be points where φ is not injective and there can be points where φ is not regular). Usually, we only have a finite number of these and we remove them to obtain coordinate charts.

<u>Examples</u>: (1) <u>Cylinders</u>:

$$\varphi(u,v) = \beta(u) + v Q$$

where Q is fixed (all the lines of the ruling are parallel)

usual cylinders: the directrix is a circle. e.g. $\beta(u) = (\cos u, \sin u, 0)$

(2) <u>Cones</u>: $\varphi(u,v) = P + v\,\delta(u)$

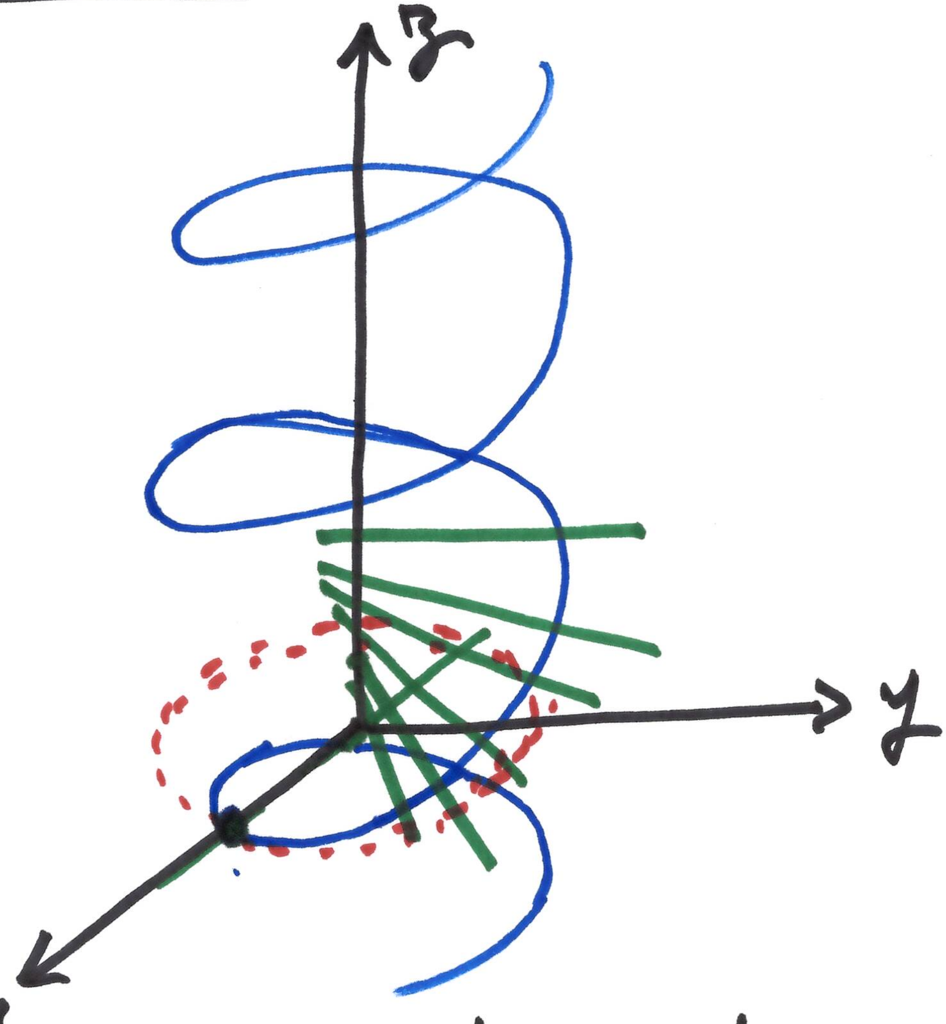
where P is ~~a~~ fixed (all the lines of the ruling pass through P).

usual cone: $\delta(u)$ is a circle.

e.g. ~~α~~ $\delta(u) = (\cos u, \sin u, 0)$.

(3) <u>Helicoids:</u>

<u>recall a helix</u>: $\alpha(u) = (a\cos u, a\sin u, bu)$



~~a~~ a helicoid is obtained as the union of horizontal lines joining a point on the helix to a point on the z-axis. a line of the ruling has parametrization: $(0,0,bu) + v(a\cos u, a\sin u, 0)$

So $\varphi(u,v) = (av\cos u, av\sin u, bu)$

$$= (0, 0, bu) + v\,(a\cos u, a\sin u, 0)$$

So $\beta(u) = (0, 0, bu)$ is the z-axis

$\delta(u) = (a\cos u, a\sin u, 0)$ is a circle.

(4) **The saddle:** $z = xy$. doubly ruled.

parametrization: $\varphi_1(u,v) = (u, v, uv)$

$$\varphi_1(u,v) = \underbrace{(u, 0, 0)}_{\beta_1(u)} + v\,\underbrace{(0, 1, u)}_{\delta_1(u)}.$$

$$\varphi_2(u,v) = (v, u, uv)$$

$$= \underbrace{(0, u, 0)}_{\beta_2(u)} + v\,\underbrace{(1, 0, u)}_{\delta_2(u)}.$$

(5) **Hyperboloids of one sheet:**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 \quad a, b, c > 0$$

$$\varphi(u,v) = \left(a v\cos u,\ b v\sin u,\ \pm c\sqrt{v^2-1}\right)$$

$$\varphi(u,v) = \left(a v\cos u,\ b v\sin u,\ \pm c\sqrt{v^2-1}\right)$$

Directrix should be ~~a circle~~ an ellipse.

assume $a=b=c=1$

Directrix is a circle. $z=0$

$$\beta(u) = (\cos u, \sin u, 0).$$

$$\varphi(u,v) = (v\cos u, v\sin u, \pm\sqrt{v^2-1})$$

$$= \beta(u) + v\,\delta(u)$$

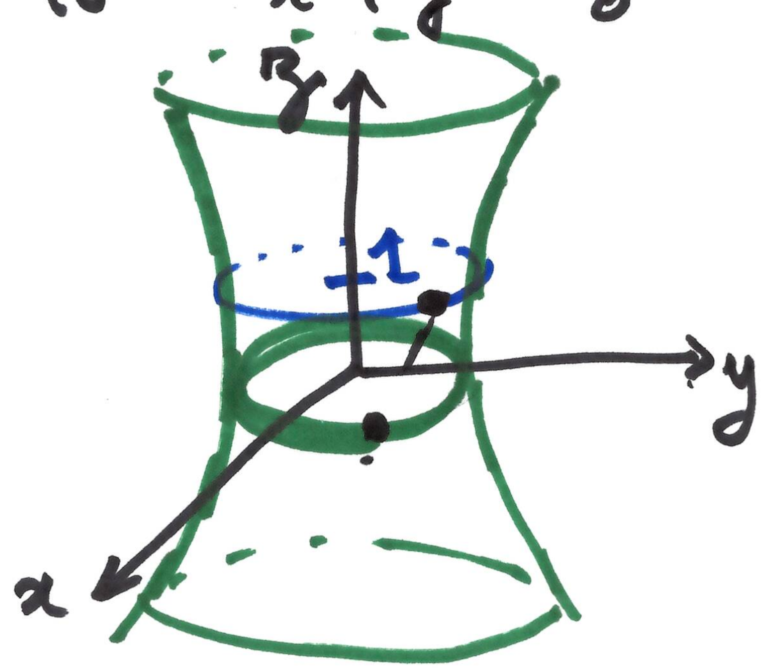
$$= (\cos u, \sin u, 0) + v\,(\delta(u))$$

$$\Rightarrow \quad v\,\delta(u) = (v\cos u, v\sin u, \pm\sqrt{v^2-1})$$

$$- v\,(\cos u, \sin u, 0)$$

$$= (\cos u\,(v-1), \sin u\,(v-1), \pm\sqrt{v^2-1})$$

Back to $x^2+y^2-z^2=1$



circle: $x^2+y^2=1 \quad z=0$

circle: $x^2+y^2=2 \quad z=1$

$\alpha_1(t_1) = (\cos t_1, \sin t_1, 0)$

$\alpha_2(t_2) = (\sqrt{2}\cos t_2, \sqrt{2}\sin t_2, 1)$

line joining the point at t_1 to ~~a~~ the point at t_2:

$$(\cos t_1, \sin t_1, 0) \cancel{+ s(\sqrt{2}\cos t_2, \sqrt{2}\sin t}$$
$$+s(\sqrt{2}\cos t_2 - \cos t_1, \sqrt{2}\sin t_2 - \sin t_1, 1)$$

$$= (\cos t_1(1-s) + \sqrt{2}\, s \cos t_2,\ \sin t_1(1-s) + \sqrt{2}\, s \sin t_2,\ s)$$

Satisfy the equation of hyperboloid:

$$\cos^2 t_1 (1-s)^2 + 2s^2\cos^2 t_2 + 2\sqrt{2}\, s(1-s)\cos t_1 \cos t_2$$
$$+ \sin^2 t_1 (1-s)^2 + 2s^2 \sin^2 t_2 + 2\sqrt{2}\, s \sin t_1 \sin t_2 (1-s)$$
$$- s^2$$

$\cancel{\varphi(u,v) = ((v^2+1)\cos u, (v^2+1)\sin u, v^2)}$.

one answer: $\beta(u) = (\cos u, \sin u, 0)$.

$\delta(u) = (-\sin u, \cos u, \pm 1)$

M a surface. U unit normal.

$S_p(V) = -\nabla_V U \qquad p \in M$

$V \in T_pM$.

if α is a curve on M with $\alpha'(0) = V$

then $S_p(V) = -\left.\dfrac{\partial U}{\partial t}\right|_0$ t = parameter for α

$$\boxed{\alpha'' \cdot U = S(\alpha') \cdot \alpha'}$$

If u is a unit vector in T_pM, then u is a principal direction when u is an eigenvector for S_p.

Two possibilities: 1) S_p has only one eigenvalue: $S_p = \lambda$ Identity. P is an umbilic point, every direction is principal.

2) S_p has two distinct eigenvalues:

k_1, k_2 : the principal curvatures.

u_1, u_2 : corresponding unit eigenvectors, the principal direction.

$u \in T_p M$ unit vector

the normal curvature in the direction of u is $S_p(u) \cdot u = k(u)$

If u is a principal direction:

$u = u_i$, then

$$k(u_i) = S_p(u_i) \cdot u_i = k_i u_i \cdot u_i = k_i$$

The Gaussian curvature is

$$K = k_1 k_2$$

The mean curvature is

$$H = \frac{1}{2}(k_1 + k_2)$$

Suppose α is a curve on M.

$\alpha' \neq 0$

Assume α is arclength parametri**z**ed

i.e., $|\alpha'| = 1$

$T = \alpha'$ the unit tangent.

U = unit normal to M

A orthonormal basis:

$T, \ U \times T, \ U$

$$\alpha'' = (\alpha'' \cdot T)T + (\alpha'' \cdot U \times T)(U \times T) + (\alpha'' \cdot U)\, U.$$