Solutions to practice problems:

(1) $\varphi(u,v) = \beta(u) + v\,\delta(u)$, $U_v = 0$

$$K = \frac{ln - m^2}{EG - F^2}$$

$\varphi_u = \beta'(u) + v\,\delta'(u)$ $\quad \varphi_v = \delta(u)$

$l = S(\varphi_u)\cdot\varphi_u$ $\quad m = S(\varphi_u)\cdot\varphi_v = S(\varphi_v)\cdot\varphi_u$

$n = S(\varphi_v)\cdot\varphi_v$

$S(\varphi_u) = -U_u$ $\quad S(\varphi_v) = -U_v = 0$

$\Rightarrow m = n = 0 \Rightarrow K = 0$

Conversely, assume $K = 0$

$l = \varphi_{uu}\cdot U$, $m = \varphi_{uv}\cdot U$, $n = \varphi_{vv}\cdot U$

$\varphi_{uu} = \beta''(u) + v\,\delta''(u)$ $\quad \varphi_{uv} = \delta'(u)$

$\varphi_{vv} = 0 \Rightarrow n = 0 \Rightarrow U_v\cdot\varphi_v = 0$

$$K = -\frac{m^2}{EG - F^2}$$ $\quad K = 0 \Rightarrow m = 0.$

$\Rightarrow S(\varphi_u)\cdot\varphi_v = S(\varphi_v)\cdot\varphi_u = 0.$

$U_v\cdot\varphi_u = 0.$

$U_v\cdot\varphi_u = U_v\cdot\varphi_v = 0$

$U_v \in T_p M$ & φ_u & φ_v form a basis of $T_p M$

$\Rightarrow U_v = 0.$ $\quad \square$

(2) Cylinders have parametrizations of the form $\varphi(u,v) = \beta(u) + vq$ where q is a constant vector.

$\varphi_u = \beta'(u) \qquad \varphi_v = q.$

$$U = \frac{\beta' \times q}{|\beta' \times q|} \Rightarrow U_v = 0.$$

Cones have parametrizations of the form $\varphi(u,v) = p + v\,\delta(u)$ where p is a fixed point.

$\varphi_u = v\,\delta'(u) \qquad \varphi_v = \delta(u)$

$$U = \frac{v\,\delta'(u) \times \delta(u)}{|v\,\delta'(u) \times \delta(u)|} = \pm \frac{\delta' \times \delta}{|\delta' \times \delta|}$$

$\Rightarrow U_v = 0.$

$U_v = \nabla_{\varphi_v} U \qquad \alpha(v) = \varphi(u_0, v)$

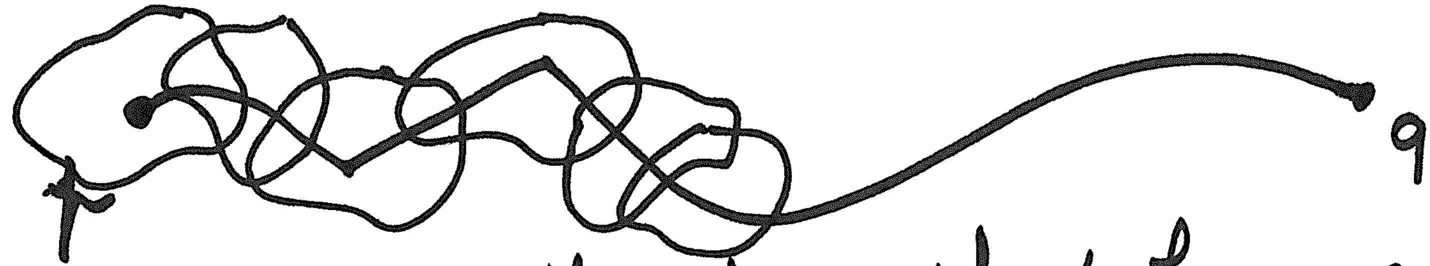
$\alpha'(v) = \varphi_v$

$$\frac{\partial}{\partial v} U(\varphi(u_0,v)) = \frac{\partial}{\partial v} U(\varphi_1(u_0,v), \varphi_2(u_0,v), \varphi_3(u_0,v))$$

$$= U_x \cdot \frac{\partial \varphi_1}{\partial v} + U_y \frac{\partial \varphi_2}{\partial v} + U_z \frac{\partial \varphi_3}{\partial \varphi_v}$$

$$= \nabla_{\varphi_v} U$$

(3) M $\quad p, q \in M$



every point on the image of α has a normal neighborhood.

the image of α from p to q is compact so we can cover it with finitely many normal neighborhoods.

$$\underset{\substack{\uparrow \\ \text{center } p \\ p = p_1}}{\mathcal{N}_{\varepsilon_1}}, \underset{\substack{\uparrow \\ \text{center } p_2}}{\mathcal{N}_{\varepsilon_2}}, \ldots, \underset{\substack{\nwarrow \\ \text{center } q \\ q = p_n}}{\mathcal{N}_{\varepsilon_n}}.$$

$\mathcal{N}_{\varepsilon_i}$ has center p_i.

Can choose $q_i \in \mathcal{N}_{\varepsilon_i} \cap \mathcal{N}_{\varepsilon_{i+1}}$

$$q_i = \alpha(t_i)$$

in $\mathcal{N}_{\varepsilon_i}$ $\exists!$ unique geodesic from p_i to q_i which is also the curve of shortest length in $\mathcal{N}_{\varepsilon_i}$.

in N_{ε_i} α = this geodesic because otherwise we could replace this piece of α with the geodesic and get a shorter curve.

Similarly from q_i to p_{i+1}.

The only places where α could have corners are the q_i's or p_i's.

We are free to move the q_i inside $N_{\varepsilon_i} \cap N_{\varepsilon_{i+1}}$ so we cannot have corners at the q_i.

we cannot have corners at the p_i because the p_i ~~are~~ is the center of N_{ε_i}.

$z \neq p_i$

$\boxed{\exists\, \varepsilon > 0 \text{ s.t. every point of distance} \leq \varepsilon \text{ from } p_i \text{ has a normal neighborhood of radius } \varepsilon.}$

Choose z on α at distance $\frac{\varepsilon}{2}$ of p_i. Choose w on α in $N_\varepsilon(z)$ on the other side of p_i from z. $w \neq p_i$

$\exists!$ geodesic from z to w in $N_\varepsilon(z)$
this geodesic is $= \alpha$ in $N_\varepsilon(z)$
because otherwize we could replace this piece of α with the geodesic and get a shorter curve.

(4) $\quad T, \; U \times T, \; U \quad$ orthonormal basis.

$$\alpha'' \cdot T = \alpha'' \cdot \frac{\alpha'}{|\alpha'|} = \frac{1}{v} \alpha'' \cdot \alpha'$$

$$v^2 = \alpha' \cdot \alpha' \Rightarrow 2 v v' = 2 \alpha' \cdot \alpha''$$

$$\Rightarrow v' = \frac{1}{v} \alpha'' \cdot \alpha' = \alpha'' \cdot T$$

$$\alpha'' \cdot (U \times T) = \alpha'' \cdot \left(U \times \frac{\alpha'}{v} \right) = \frac{1}{v} \alpha'' \cdot (U \times \alpha')$$

$$= \frac{1}{v} U \cdot (\alpha' \times \alpha'')$$

$\alpha' \times \alpha'' \,//\, B.$
$\theta =$ angle between B and U.

$$= \frac{1}{v} |\alpha' \times \alpha''| \cos\theta .$$

$$= \frac{1}{v} \kappa v^3 \cos\theta = \kappa v^2 \cos\theta$$

$$= \kappa_g \, v^2$$

$\square$

Consider the sphere $S^2: x^2+y^2+z^2=1$

$p = (0,0,1)$.

identify $T_p S^2$ with the x,y plane

$v \in T_p S^2 = \mathbb{R}^2 \qquad v = (a,b)$

$\exp_p(v) = ? \qquad \sqrt{a^2+b^2} = 1$

$= \gamma_v(1)$

$\gamma_v =$ geodesic s.t. $\gamma_v(0) = p$.

$\gamma_v'(0) = v$

The ~~plo~~ vertical plane containing v has equation $bx - ay = 0$.

intersect with S^2: $\qquad y = \frac{b}{a}x$.

$$x^2 + \frac{b^2}{a^2}x^2 + z^2 = 1 \qquad y = \frac{b}{a}x$$

$$\frac{a^2+b^2}{a^2}x^2 + z^2 = 1 \qquad y = \frac{b}{a}x.$$

parametrization:

$$\gamma_v(t) = \left(\frac{a}{\sqrt{a^2+b^2}} \cancel{\sin t}\, \sin t, \frac{b}{\sqrt{a^2+b^2}} \cancel{\sin t}\, \sin t, \cancel{\sin t}\, \cos t\right)$$

$\Rightarrow \gamma_v(0) = (0,0,1)$.

$$\gamma_v'(0) = \left(\frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}, 0\right) = \frac{v}{\sqrt{a^2+b^2}}$$

$\sqrt{a^2+b^2} = 1$

$$\gamma_v(t) = (a \sin t, b \sin t, \cos t)$$

$$\exp_p(a, b) = \cancel{(a \sin t, b \sin t, \cos t)}$$

when $\sqrt{a^2+b^2} = 1$ $\quad (a \sin 1, b \sin 1, \cos 1)$

$$\boxed{\exp_p(ta, tb) = (a \sin t, b \sin t, \cos t)}$$

$$\exp_p(v) = \gamma_v(1)$$

$$\exp_p(tv) = \gamma_v(t)$$