~~$\exp_p(v) = \gamma_v(\sqrt{a^2+b^2})$~~

~~$\gamma_v = (a \sin t,\ b \cos t,\ \cos t)$~~

<u>Def</u>: An abstract surface is a set S with a collection of coordinate charts such that:

(a coordinate chart is a 1-to-1 map from a domain in $\mathbb{R}^2$ to S)

1) S = union of the images of the coordinate charts.

2) $\forall$ choice of two charts

$$\varphi_1 : D_1 \hookrightarrow S \qquad \varphi_2 : D_2 \hookrightarrow S$$

$\varphi_1^{-1}(\text{image}(D_1) \cap \text{image}(D_2))$ is open in $\mathbb{R}^2$.

and $\boxed{\varphi_2^{-1}\varphi_1}$ is smooth from D_1 to D_2

in fact from $\varphi_1^{-1}(\text{image}(D_1) \cap \text{image}(D_2))$

to $\varphi_2^{-1}($ " $)$

3) (Hausdorff axiom). $\forall\ p \neq q \in S$

$\exists$ coordinate charts φ_1, φ_2 s.t.

$p \in \text{Im}\,\varphi_1$, $p \notin \text{Im}\,\varphi_2$.
$q \notin \text{Im}\,\varphi_1$, $q \in \text{Im}\,\varphi_2$
$\text{Im}\,\varphi_1 \cap \text{Im}\,\varphi_2 = \phi$

<u>Remark</u>: Can define a topology on S:
open sets = unions of images of open sets from domains of charts.

We want a way of measuring distance on S. We compute lengths of curves by integrating the length of their velocity vectors.

Given a chart $\varphi : D \to S$.
the tangent plane to S at p
is $T_{\varphi^{-1}(p)} D = T_p S$ | or velocity vectors of curves in S.

We need to compute lengths of tangent vectors to S.

When ~~the~~ $S \subset \mathbb{R}^3$, $|v| = \sqrt{v \cdot v}$.

We define metrics on abstract surfaces via inner products.

Def: An inner product on $\mathbb{R}^2$ is a map

$$\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$$
$$(v, w) \longmapsto \langle v, w \rangle$$

s.t.

1) $\langle \, , \, \rangle$ is bilinear (linear in v and in w)

2) $\langle \, , \, \rangle$ is symmetric, i.e.:
$$\langle v, w \rangle = \langle w, v \rangle \quad \forall \, v, w \in \mathbb{R}^2$$

3) $\forall \, v \quad \langle v, v \rangle \geq 0$
and $\langle v, v \rangle = 0 \iff v = 0$.

Remark: If symmetry fails, define
$$\langle v, w \rangle_2^* = \langle v, w \rangle + \langle w, v \rangle.$$

Def: Given an inner product, we can define a length or norm on $\mathbb{R}^2$ via $\|v\| := \sqrt{\langle v, v \rangle}$.

<u>Def</u>: given $v, w \in \mathbb{R}^2 \setminus \{0\}$.
can define the angle between v & w
as $\arccos\left(\frac{\langle v, w\rangle}{\|v\|\,\|w\|}\right)$

This is well-defined because we have

<u>Lemma</u>: (Schwarz' inequality):

$$|\langle v, w\rangle| \leq \|v\|\,\|w\|$$

<u>Proof</u>: $0 \leq \langle v + tw, v + tw\rangle$

$$= \langle v, v\rangle + \langle v, tw\rangle + \langle tw, v\rangle + \langle tw, tw\rangle$$

$$= \underbrace{\langle v, v\rangle}_{\in \mathbb{R}} + 2t\underbrace{\langle v, w\rangle}_{\in \mathbb{R}} + t^2\underbrace{\langle w, w\rangle}_{\in \mathbb{R}}.$$

degree 2 polynomial is always ≥ 0

$\Leftrightarrow$ at most one real root.

$\Leftrightarrow$ discriminant ≤ 0.

$\Leftrightarrow$ $\langle v, w\rangle^2 - \langle w, w\rangle\langle v, v\rangle \leq 0$

$\Leftrightarrow$ $|\langle v, w\rangle| \leq \|v\|\,\|w\|$ $\square$

Fact: The datum of an inner product $\Longleftrightarrow$ the datum of a symmetric invertible 2×2 matrix with positive eigenvalues.

Given $\langle\ ,\ \rangle$ on $\mathbb{R}^2 \ni (1,0)$, $(0,1)$ define the matrix $A = (a_{ij})$.

$a_{ij} = \langle e_i , e_j \rangle \qquad e_1 = (1,0),\ e_2 = (0,1)$

Given $v = (x_1, y_1) \quad w = (x_2, y_2)$.

$$\begin{aligned}\langle v, w\rangle &= \langle x_1 e_1 + y_1 e_2 , x_2 e_1 + y_2 e_2 \rangle \\ &= x_1 x_2 a_{11} + x_1 y_2 a_{12} + x_2 y_1 a_{21} \\ &\qquad + x_2 y_2 a_{22} \\ &= (x_1\ y_1) \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\ &= {}^t v\, A\, w\end{aligned}$$

We use a family of inner products to define a metric on an abstract surface.

<u>Definition</u>: Given an abstract surface S, a vector field V on S is the datum for each $p \in S$ of a vector $V(p) \in T_p S$.

We say V is smooth if V varies smoothly with p. This means $\forall$ coordinate chart $\varphi : D \to S$

$\cancel{V(p) = V(\varphi(p))}$ ~~is a smooth functions~~ $\cancel{= (f(\varphi(p)), g(\varphi(p)))}$

$p \in \mathrm{Im}\, \varphi$.

$$T_{\varphi^{-1}(p)}D \ni W(\varphi^{-1}(p)) = V(p) \in T_p S$$

$$T_{\varphi^{-1}(p)}D = \mathbb{R}^2$$

$$\varphi^{-1}(p) = (x, y).$$

$$W(x,y) = (f(x,y), g(x,y))$$

$$f, g \ C^\infty.$$

<u>Def</u>: Given an abstract surface S, a metric on S is the data, for all $p \in S$ of an inner product $\langle\, , \rangle_p$.

on $T_pS \cong \mathbb{R}^2$

which varies smoothly with p.

This means $\forall$ C^∞ vector fields V, W on S

$\langle V(p), W(p) \rangle_p$ is a C^∞ function

$\langle \, , \, \rangle_p = g(p)$ is called the metric tensor.

<u>Def</u>: A geometric surface is an abstract surface with a metric.

<u>Construction method 1:</u> conformal change.

One way of defining a new metric on a domain $D \subset \mathbb{R}^2$ is by rescaling the Euclidean length of vectors.

given a C^∞ function h on D s.t. $h(p) > 0$ $\forall$ $p \in D$, define

$$\langle v, w \rangle_{h,p} := \frac{v \cdot w}{h^2(p)} \quad v, w \in T_pD$$

This is called a conformal metric, h is the scaling factor or ruler function.

<u>Example</u> ① The Poincaré plane P:

$$h(x,y) = y$$

P = the upper half plane $y>0$ with metric

$$\langle v, w\rangle_{(x,y)} = \frac{v\cdot w}{y^2}$$

$$\varphi : D = P \longrightarrow P$$

Identity.

$$\varphi_x = (1,0) \qquad \varphi_y = (0,1)$$

$$E = \varphi_x \cdot \varphi_x = \langle \varphi_x, \varphi_x\rangle_{(x,y)} = \frac{1}{y^2}$$

$$F = \varphi_x \cdot \varphi_y = \langle \varphi_x, \varphi_y\rangle_{(x,y)} = 0$$

$$G = \frac{1}{y^2} \qquad \rightarrow \qquad A = \begin{pmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{pmatrix}$$

<u>Example</u> ② The Poincaré disk model of the hyperbolic plane.

$H =$ open disk of radius 2.

$$\langle v, w\rangle_p = \frac{v\cdot w}{\left(1-\frac{x^2+y^2}{4}\right)^2}$$

$p=(x,y)$

$$h(x,y) = 1-\frac{x^2+y^2}{4} > 0$$

<u>Construction method 2</u>: pull-back.

Suppose N is a geometric surface and $F: M \longrightarrow N$ is smooth and regular (M an abstract surface).

(regular means $F_*: T_pM \to T_{F(p)}N$ is a linear isom. $\forall p$)

If $\langle\, ,\rangle_N$ is the metric on N, define $\langle\, ,\rangle_M$ via

$$\langle v, w\rangle_M := \langle F_* v, F_* w\rangle_N .$$

<u>Example</u>: The stereographic sphere:

S^2 unit sphere in $\mathbb{R}^3$ $N=(0,0,1)$

$S^2_N := S^2 \setminus \{N\}$

St: $S^2_N \longrightarrow \mathbb{R}^2 = x,y$ plane.

$p \longmapsto$ point of intersection ~~of~~ of the x,y plane with the line through p & N.