$St: S^2_N \longrightarrow \mathbb{R}^2 = x,y$ plane.

$p \longmapsto$ point of intersection of the x,y plane with the line through p & N.

The stereographic metric on S^2_N is the pull-back of the Euclidean metric by St.

$$p = (\cos u \cos v, \sin u \cos v, \sin v)$$

$$N = (0,0,1).$$

line through p & N:

$$\alpha_p(t) = (0,0,1) + t(\cos u \cos v, \sin u \cos v, \sin v - 1)$$

$\cap$ (x,y) plane $\Rightarrow z = 0$

So $1 + t(\sin v - 1) = 0 \Rightarrow t = \dfrac{1}{1-\sin v}$

point $St(p) = \left(\dfrac{\cos u \cos v}{1-\sin v}, \dfrac{\sin u \cos v}{1-\sin v}, 0\right)$

Compute the differential of St at p:

$$St_*(\varphi_u) = \frac{\partial}{\partial u}\left(St(\varphi(u,v))\right)$$

$$= \left(\frac{-\sin u \cos v}{1-\sin v}, \frac{\cos u \cos v}{1-\sin v}, 0\right)$$

$$St_*(\varphi_v) = \frac{\partial}{\partial v}\left(St(\varphi(u,v))\right)$$

$$= \left(\frac{-\cos u \sin v}{1-\sin v}, \frac{-\sin u \sin v}{1-\sin v}, 0\right)$$

$$+ \frac{\cos v}{(1-\sin v)^2}\left(\cos u \cos v, \sin u \cos v, 0\right)$$

$$E = \langle St_*(\varphi_u), St_*(\varphi_u)\rangle_{\mathbb{R}^2}$$

$$F = \langle St_*(\varphi_u), St_*(\varphi_v)\rangle_{\mathbb{R}^2}$$

$$G = \langle St_*(\varphi_v), St_*(\varphi_v)\rangle_{\mathbb{R}^2}$$

<u>Example:</u> <u>The stereographic plane:</u>

$$St : S^2_N \longrightarrow \mathbb{R}^2$$

$$St^{-1} : \mathbb{R}^2 \longrightarrow S^2_N.$$

The stereographic plane is $\mathbb{R}^2$ with metric the pull-back of the ~~Euclidean~~ usual metric (from $\mathbb{R}^3$) by St^{-1}.

<u>Homework</u>: Compute this as above.

Construction method 3:

Can define the metric via the metric coefficients E, F, G.

First choose a coordinate chart

$$\varphi : D \hookrightarrow M. \quad T_{\varphi^{-1}(p)} D = T_p M$$

φ_u = image of $(1,0)$

φ_v = " " $(0,1)$.

Given three smooth functions E, F, G on D s.t. $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ defines an inner product at every point, we obtain a metric.

Need $\begin{pmatrix} E & F \\ F & G \end{pmatrix}$ invertible with positive eigenvalues at every point.

Differential forms:

① Differential 1-forms.

Def: M a surface

A C^∞ differential 1-form ω on M is the datum, for every $p \in M$, of a linear form ω_p on T_pM s.t. these forms vary smoothly with p.

This means $\forall$ C^∞ vector field V on M, the function $\omega_p(V(p))$ is C^∞ (or smooth)..

<u>Example:</u> $M = \mathbb{R}^2$, then $T_pM = \mathbb{R}^2$ $\forall p$.

$e_1 = (1,0)$, $e_2 = (0,1)$.

define dx to be the differential form which at every p sends $(1,0)$ to 1 and $(0,1)$ to 0.

define dy to be " " " " " " $(1,0)$ to 0 and $(0,1)$ to 1.

These are constant differential forms. Any other differential form on $\mathbb{R}^2$ is of the form $f\,dx + g\,dy$ where f, g are C^∞ on $\mathbb{R}^2$.

take $p \in \mathbb{R}^2$, $v \in T_p\mathbb{R}^2 = \mathbb{R}^2$

$p = (x, y)$. $\quad v = (\lambda, \mu)$

then $(f\,dx + g\,dy)_p(v) = f(p)\lambda + g(p)\mu$

$$(f\,dx + g\,dy)_p = f(p)\,dx + g(p)\,dy.$$

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given a function f on M, we can a differential 1-form df:

the differential or exterior derivative

<u>Def</u>: M surface, f C^∞ function on M. df is the differential form defined as follows: $\forall p \in M$, $v \in T_pM$ choose a curve α in M s.t.

$$\alpha(0) = p \qquad \alpha'(0) = v$$

then
$$(df)_p(v) := \frac{d}{dt}\left(f(\alpha(t))\right)\Big|_{t=0}$$
$$:= \nabla_v f$$

<u>Example</u>: $M = \mathbb{R}^2$, $f : \mathbb{R}^2 \to \mathbb{R}$ C^∞ function.

$p=(x,y)\in\mathbb{R}^2$ $\quad v=(\lambda,\mu)\in T_p\mathbb{R}^2=\mathbb{R}^2$

~~$(df)_p(v)=$~~ Choose $\alpha(t)=(\lambda t+x,\ \mu t+y)$

$$(df)_p(v)=\frac{d}{dt}\Big(f(\lambda t+x,\ \mu t+y)\Big)\Big|_{t=0}$$

$$=\lambda\frac{\partial f}{\partial x}(p)+\mu\frac{\partial f}{\partial y}(p)=\nabla_v f(p)$$

$$=\frac{\partial f}{\partial x}(p)\,dx(v)+\frac{\partial f}{\partial y}(p)\,dy(v).$$

$$\Rightarrow\quad (df)_p=\frac{\partial f}{\partial x}(p)\,dx+\frac{\partial f}{\partial y}(p)\,dy$$

$$\Rightarrow\quad df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy.$$

② Differential 2-forms:

Def: M a surface. A differential 2-form ω is the datum for every $p\in M$ of an alternating ~~[illegible]~~ bilinear form ω_p on T_pM which varies smoothly with p.

This means: $\forall\ p \in M$,

we have a map $T_pM \times T_pM \to \mathbb{R}$

$$(v, w) \longmapsto \omega_p(v,w)$$

$\omega_p(v,w)$ linear in v and w.

and $\omega_p(w,v) = -\omega_p(v,w)$ $\forall\ v, w$.

$\forall\ C^\infty$ vector fields V, W on M

$p \mapsto \omega_p(V(p), W(p))$ is a C^∞ function

Remark: given two linear forms $l, m : \mathbb{R}^2 \to \mathbb{R}$, we can define an alternating form $l \wedge m$ on $\mathbb{R}^2$ as follows:

$$(l \wedge m)(v, w) = l(v)\, m(w) - l(w)\, m(v)$$

<u>Def</u>: given two 1-forms ω, θ on M, we can define the 2-form $\omega \wedge \theta$

as follows: $p \in M$ $\quad v, w \in T_pM$

$$(\omega \wedge \theta)_p (v, w) := \omega_p(v)\theta_p(w) - \omega_p(w)\theta_p(v)$$

~~Example: $M = \mathbb{R}^2$~~

Notation: $\Omega^1(M) :=$ the vector space of 1-forms on M

$\Omega^2(M) :=$ the vector space of 2-forms on M.

$\Omega^0(M) :=$ the vector space of C^∞ functions on M.

Example: $M = \mathbb{R}^2$

$$\Omega^1(M) = \Omega^0(M)\, dx + \Omega^0(M)\, dy$$

$p \in \mathbb{R}^2$ $\quad p = (x, y)$ $\quad T_p\mathbb{R}^2 = \mathbb{R}^2$.

$v = (\lambda, \mu)$ $\qquad w = (\nu, \tau)$

$= \lambda e_1 + \mu e_2$ $\qquad = \nu e_1 + \tau e_2$.

ω alternating form on $\mathbb{R}^2$:

$$\omega(v,w) = \omega(\lambda e_1 + \mu e_2,\ \nu e_1 + \tau e_2)$$

$$= \nu\lambda\omega(e_1,e_1) + \lambda\tau\omega(e_1,e_2)$$

$$+ \mu\nu\omega(e_2,e_1) + \mu\tau\omega(e_2,e_2)$$

$\omega(e_1,e_1) = -\omega(e_1,e_1) \Rightarrow = 0$

also $\omega(e_2,e_2) = 0$

$\omega(e_1,e_2) = -\omega(e_2,e_1)$

$$\omega(v,w) = (\lambda\tau - \mu\nu)\,\underline{\omega(e_1,e_2)},$$

$$= \det\left(\begin{array}{cc} \boxed{\begin{matrix}\lambda\\ \mu\end{matrix}} & \boxed{\begin{matrix}\nu\\ \tau\end{matrix}} \\ v & w \end{array}\right)\ \omega(e_1,e_2)$$

We just proved:

<u>Prop</u>: $\forall$ alternating form ω on $\mathbb{R}^2$, $\exists$ a scalar l s.t. $\omega = l\,\underline{\det}$.

$$dx \wedge dy\,(v,w) = dx(v)\,dy(w) - dx(w)\,dy(v)$$

$$= \lambda\tau - \nu\mu$$

$$= \underline{\underline{\det}}.(v,w).$$

Any C^∞ 2-form on $\mathbb{R}^2$ can be written as $f\,dx\wedge dy$ where f is a C^∞ function on $\mathbb{R}^2$.

So $\Omega^2(\mathbb{R}^2) = \Omega^0(\mathbb{R}^2)\,dx\wedge dy$.