Any C^∞ 2-form on $\mathbb{R}^2$ can be written as $f\, dx\wedge dy$ where f is a C^∞ function on $\mathbb{R}^2$.

So $\Omega^2(\mathbb{R}^2) = \Omega^0(\mathbb{R}^2)\, dx\wedge dy$.

~~Other~~ On any surface M:

Frame fields:

Def: A frame field on M is the data for all $p\in M$ of an orthonormal basis of T_pM.

Usually denoted: $\{e_1, e_2\}$.

e_i vector field on M.

$$\langle e_i(p), e_j(p)\rangle_p = \delta_{ij} \begin{cases} = 1 \text{ if } i=j \\ = 0 \text{ if } i\neq j. \end{cases}$$

Def: A frame field on $\mathbb{R}^3$ is the data for all $p\in\mathbb{R}^3$ of an orthonormal basis of $T_p\mathbb{R}^3 = \mathbb{R}^3$. (~~e.g.~~ Frenet frames were adapted to curves)

<u>Def</u>: Given a frame field, we can define the dual 1-forms $\{\theta_1, \theta_2\}$:
$\forall p$ $\{\theta_1(p), \theta_2(p)\}$ = dual basis of $\{e_1(p), e_2(p)\}$.

If ω is a 1-form on M, then we can write $\omega = f\theta_1 + g\theta_2$ where f, g are C^∞ functions on M.

<u>In fact</u>: $\omega = \omega(e_1)\theta_1 + \omega(e_2)\theta_2$

$$v \in T_pM = \mathbb{R}e_1(p) + \mathbb{R}e_2(p)$$

$$v = \lambda e_1(p) + \mu e_2(p)$$

$$\omega_p(v) = \omega(\lambda e_1(p) + \mu e_2(p))$$
$$= \lambda\omega(e_1(p)) + \mu\,\omega(e_2(p)).$$

$$\theta_1(v) = \theta_1(\lambda e_1(p) + \mu e_2(p)) = \lambda$$
$$\theta_2(v) = \theta_2(\quad \text{"} \quad) = \mu$$

$$\omega_p(v) = \theta_1(v)\,\omega(e_1(p)) + \theta_2(v)\,\omega(e_2(p))$$
$$\Rightarrow \omega_p = \omega_p(e_1(p))\,\theta_1 + \omega_p(e_2(p))\,\theta_2.$$

$(\theta_1 \wedge \theta_2)_p$ is an alternating bilinear form on T_pM $\forall p$

$\theta_1 \wedge \theta_2 \in \Omega^2(M)$.

$\omega \in \Omega^2 M \qquad w, v \in T_pM$

$$v = \lambda e_1 + \mu e_2$$
$$w = \nu e_1 + \tau e_2.$$

$$(\theta_1 \wedge \theta_2)(v, w) = \theta_1(v)\theta_2(w) - \theta_1(w)\theta_2(v)$$
$$= (\lambda\tau - \mu\nu).$$

$$\omega(v, w) = (\lambda\tau - \mu\nu)\,\omega(e_1, e_2)$$

So $\quad \omega = \omega(e_1, e_2)\, \theta_1 \wedge \theta_2.$

<u>Differentials of 1-forms:</u>

<u>Def</u>: ω a 1-form on M.

$d\omega$ is ~~a~~ the 2-form on M s.t.

$\forall$ coordinate chart $\varphi : D \to M$.

$$(d\omega)(\varphi_u, \varphi_v) = \frac{\partial}{\partial u}(\omega(\varphi_v)) - \frac{\partial}{\partial v}(\omega(\varphi_u))$$

<u>Example</u>: $M = \mathbb{R}^2$. $\varphi : \mathbb{R}^2 \xrightarrow[\text{Identity}]{} \mathbb{R}^2$.

$\omega = f\,dx + g\,dy$

$\varphi_x = (1, 0) \qquad \varphi_y = (0, 1)$

$\therefore\ d\omega(\varphi_x, \varphi_y) = \frac{\partial}{\partial x}(g) - \frac{\partial}{\partial y}(f)$

$= \frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}$.

$w_1 = (a, b) \qquad w_2 = (c, d)$.

$\underbrace{d\omega}(w_1, w_2) = d\omega(\varphi_x, \varphi_y) \cdot \det(w_1, w_2)$

$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy$.

~~Similarly, on M with frame field $\{e_1, e_2\}$, dual basis $\{\theta_1, \theta_2\}$~~

<u>Example</u>: differentials and frame fields on $\mathbb{R}^3$: $\{e_1, e_2, e_3\}$.

$v \in \mathbb{R}^3 \qquad \nabla_v e_i \qquad p \in \mathbb{R}^3 : e_i(p)$

<u>at p</u>: $\nabla_v e_1 = \omega_{11}(v) e_1 + \omega_{12}(v) e_2 + \omega_{13}(v) e_3$

$\nabla_v e_2 = \omega_{21}(v) e_1 + \omega_{22}(v) e_2 + \omega_{23}(v) e_3$

$\nabla_v e_3 = \omega_{31}(v) e_1 + \omega_{32}(v) e_2 + \omega_{33}(v) e_3$

∇_v is linear in v

$\Rightarrow$ $(\omega_{ij})_p(v)$ are linear in v.

$\Rightarrow$ $(\omega_{ij})_p$ is a linear form on $\mathbb{R}^3 = T_p\mathbb{R}^3$.

$\Rightarrow$ ω_{ij} is a 1-form on $\mathbb{R}^3$.

<u>Def</u>: These are the connection forms for $\{e_1, e_2, e_3\}$.

$$e_1 \cdot e_1 = 1$$

$$\Rightarrow \nabla_v (e_1 \cdot e_1) = 0 \qquad \forall v \in \mathbb{R}^3$$

$$2 \nabla_v e_1 \cdot e_1 = 0$$

$$\nabla_v e_1 = \omega_{11}(v) e_1 + \omega_{12}(v) e_2 + \omega_{13}(v) e_3$$

$$0 = \nabla_v e_1 \cdot e_1 = \omega_{11}(v) \quad \forall v$$

$$\Rightarrow \omega_{11} = 0$$

Similarly $\omega_{22} = \omega_{33} = 0$.

$$e_1 \cdot e_2 = 0 \Rightarrow \nabla_v (e_1 \cdot e_2) = 0$$

$$= \nabla_v e_1 \cdot e_2 + \nabla_v e_2 \cdot e_1 = 0$$

$$= \omega_{12}(v) + \omega_{21}(v) = 0$$

$$\forall v \quad \omega_{12}(v) = -\omega_{21}(v).$$

$$\Rightarrow \quad \omega_{21} = -\omega_{12}.$$

Similarly $\omega_{13} = -\omega_{31}$, $\omega_{23} = -\omega_{32}$

$$\begin{pmatrix} \nabla_v e_1 \\ \nabla_v e_2 \\ \nabla_v e_3 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

The dual frame:

$\{e_1, e_2, e_3\}$ $\quad \mathbb{R}^3 : \left\{ \begin{array}{l} (1,0,0) \\ (0,1,0) \\ (0,0,1) \end{array} \right\}$

$e_1 = (e_{11}, e_{12}, e_{13})$

$e_2 = (e_{21}, e_{22}, e_{23})$

$e_3 = (e_{31}, e_{32}, e_{33})$

dual basis to $\{(1,0,0), (0,1,0), (0,0,1)\}$:

$\{dx, dy, dz\}$

$v = \lambda e_1 + \mu e_2 + \nu e_3.$

$\theta_1(v) = \lambda, \quad \theta_2(v) = \mu, \quad \theta_3(v) = \nu.$

$$\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} = A$$

$$\begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} \theta_{11} & \theta_{12} & \theta_{13} \\ \theta_{21} & \theta_{22} & \theta_{23} \\ \theta_{31} & \theta_{32} & \theta_{33} \end{pmatrix} = B.$$

$$B\,{}^tA = \mathrm{Id}. \Rightarrow B = {}^tA^{-1}$$

$\{e_1, e_2, e_3\}$ is an orthonormal basis so $A\,{}^tA = \mathrm{Id} \Rightarrow A = {}^tA^{-1}$

So $A = B$. at every point.

$$\Rightarrow \quad \theta_1 = e_{11}\,dx + e_{12}\,dy + e_{13}\,dz.$$
$$\theta_2 = e_{21}\,dx + e_{22}\,dy + e_{23}\,dz$$
$$\theta_3 = e_{31}\,dx + e_{32}\,dy + e_{33}\,dz.$$

Assume $M \subset \mathbb{R}^3$ is a surface.
assume $\{e_1, e_2, e_3\}$ is adapted to M, meaning at every $p \in M$ $\{e_1, e_2\}$ is a basis of T_pM and e_3 is a unit normal to M.
$\{\theta_1, \theta_2\}$ are then 1-forms on M.