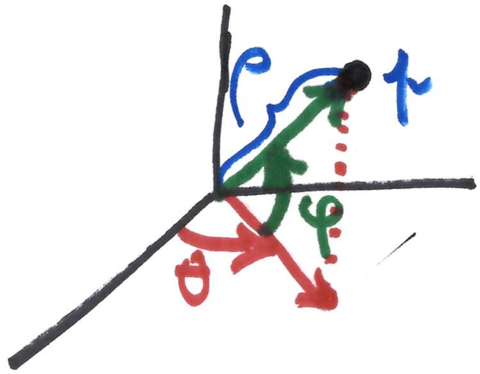
Example 2 for Frame fields:

Spherical frame field in $\mathbb{R}^3$:

spherical coordinates : (ρ, θ, φ).



E_1 : unit vector pointing in the direction where ρ increases.

E_2 : unit vector pointing in the direction where θ increases. (ρ, φ are fixed).

E_3 : unit vector pointing in the direction where φ increases (ρ, θ are fixed)

$$E_2 = -\sin\theta\, U_1 + \cos\theta\, U_2.$$

$U_1 = (1, 0, 0)$ $\quad$ $U_2 = (0, 1, 0)$

$U_3 = (0, 0, 1)$

$E_1 \,//\,$ vector 0 to p.

$p = (\rho\cos\theta\cos\varphi, \rho\sin\theta\cos\varphi, \rho\sin\varphi)$

$|p| = \rho$.

$E_1 = \cancel{\rho}\cos\theta\cos\varphi\, U_1 + \sin\theta\cos\varphi\, U_2 + \sin\varphi\, U_3$.

$E_3 = \dfrac{\partial E_1}{\partial\varphi} = -\cos\theta\sin\varphi\, U_1 - \sin\theta\sin\varphi\, U_2 + \cos\varphi\, U_3$.

$\theta_1, \theta_2, \theta_3$ dual basis of E_1, E_2, E_3.

The matrix of $\theta_1, \theta_2, \theta_3$ on the basis dx, dy, dz is equal to the matrix of E_1, E_2, E_3 on U_1, U_2, U_3:

$$U_1 = \frac{\partial}{\partial x}, \quad U_2 = \frac{\partial}{\partial y}, \quad U_3 = \frac{\partial}{\partial z}$$

$\theta_1 = \cos\theta\cos\varphi\, dx + \sin\theta\cos\varphi\, dy + \sin\varphi\, dz$.

$\theta_2 = -\sin\theta\, dx + \cos\theta\, dy$

$\theta_3 = -\cos\theta\sin\varphi\, dx - \sin\theta\sin\varphi\, dy + \cos\varphi\, dz$

$$x = \rho \cos\theta \cos\varphi$$
$$y = \rho \sin\theta \cos\varphi$$
$$z = \rho \sin\varphi.$$

$$dx = \cos\theta \cos\varphi \, d\rho - \rho \sin\theta \cos\varphi \, d\theta - \rho \cos\theta \sin\varphi \, d\varphi$$

$$dy = \sin\theta \cos\varphi \, d\rho + \rho \cos\theta \cos\varphi \, d\theta - \rho \sin\theta \sin\varphi \, d\varphi$$

$$dz = \sin\varphi \, d\rho + \rho \cos\varphi \, d\varphi$$

substitute dx, dy, dz in $\theta_1, \theta_2, \theta_3$:

$$\theta_1 = d\rho \qquad \theta_2 = \rho \cos\varphi \, d\theta$$
$$\theta_3 = \rho \, d\varphi$$

$$d\theta_1 = d(d\rho) = 0$$
$$d\theta_2 = d(\rho \cos\varphi \, d\theta) = d(\rho\cos\varphi) \wedge d\theta$$
$$= \cos\varphi \, d\rho \wedge d\theta - \rho \sin\varphi \, d\varphi \wedge d\theta.$$
$$= \cos\varphi \, d\rho \wedge d\theta + \rho \sin\varphi \, d\theta \wedge d\varphi.$$
$$d\theta_3 = d\rho \wedge d\varphi.$$

Connections forms: $\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix}$

$$\Omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}$$

Structural equations:

$$d\Theta = \Omega \wedge \Theta$$

$$d\Omega = \Omega \wedge \Omega$$

$$\forall v: \quad \nabla_v E_1 = \omega_{12}(v) E_2 + \omega_{13}(v) E_3.$$

$$\frac{\partial}{\partial \rho} E_1 = \frac{\partial}{\partial \rho} E_2 = \frac{\partial}{\partial \rho} E_3 = 0$$

$$\frac{\partial}{\partial \theta} E_1 = -\sin\theta \cos\varphi\, U_1 + \cos\theta \cos\varphi\, U_2$$
$$= \cos\varphi\, E_2$$

$$\frac{\partial}{\partial \varphi} E_1 = -\cos\theta \sin\varphi\, U_1 - \sin\theta \sin\varphi\, U_2 + \cos\varphi\, U_3$$
$$= E_3.$$

$$\frac{\partial}{\partial\theta} E_2 = -\cos\theta\, U_1 - \sin\theta\, U_2$$

$$\cancel{= \frac{1}{\sin\varphi} E_3 - \frac{\cos\varphi}{\sin\varphi} U_3}.$$

$$A = \begin{pmatrix} \cos\theta\cos\varphi & \sin\theta\cos\varphi & \sin\varphi \\ -\sin\theta & \cos\theta & 0 \\ -\cos\theta\sin\varphi & -\sin\theta\sin\varphi & \cos\varphi \end{pmatrix}$$

$$\begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = A \begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix}$$

$A\,{}^tA = I$ orthogonal matrix.

$$\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix} = A^{-1} \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix} = {}^tA \begin{pmatrix} E_1 \\ E_2 \\ E_3 \end{pmatrix}$$

$$\Rightarrow \quad U_1 = \cos\theta\cos\varphi\, E_1 - \sin\theta\, E_2 - \cos\theta\sin\varphi\, E_3$$

$$U_2 = \sin\theta\cos\varphi\, E_1 + \cos\theta\, E_2 - \sin\theta\sin\varphi\, E_3.$$

$$\Rightarrow \frac{\partial}{\partial\theta} E_2 = -\cos\varphi E_1 + \sin\varphi E_3.$$

$$\frac{\partial}{\partial\varphi} E_2 = 0$$

$$\frac{\partial}{\partial\theta} E_3 = \sin\theta \sin\varphi U_1 - \cos\theta \sin\varphi U_2$$
$$= -\sin\varphi E_2$$

$$\frac{\partial}{\partial\varphi} E_3 = -\cos\theta \cos\varphi U_1 - \sin\theta \cos\varphi U_2 - \sin\varphi U_3.$$
$$= - E_1 \qquad \cos\varphi\, d\theta \qquad d\varphi$$

$$\nabla_v E_1 = \cancel{\cos\varphi\, E_2} + \cos\varphi\, E_2 + E_3$$

$$\nabla_v E_2 = -\cos\varphi\, d\theta\, E_1 + \sin\varphi\, d\theta\, E_3$$

$$\nabla_v E_3 = -d\varphi\, E_1 - \sin\varphi\, d\theta\, E_2.$$

$$\Omega = \begin{pmatrix} 0 & \cos\varphi\, d\theta & d\varphi \\ -\cos\varphi\, d\theta & 0 & \sin\varphi\, d\theta \\ -d\varphi & -\sin\varphi\, d\theta & 0 \end{pmatrix}$$

$$d\Theta = \begin{pmatrix} 0 \\ \cos\varphi \, d\rho \wedge d\theta + \rho \sin\varphi \, d\theta \wedge d\varphi \\ d\rho \wedge d\varphi. \end{pmatrix}$$

$$\Omega \wedge \Theta = \begin{pmatrix} 0 & \cos\varphi \, d\theta & d\varphi \\ -\cos\varphi \, d\theta & 0 & \sin\varphi \, d\theta \\ -d\varphi & -\sin\varphi \, d\theta & 0 \end{pmatrix} \begin{pmatrix} d\rho \\ \rho\cos\varphi \, d\theta \\ \rho \, d\varphi \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ -\cos\varphi \, d\theta \wedge d\rho + \rho \sin\varphi \, d\theta \wedge d\varphi \\ - d\varphi \wedge d\rho \end{pmatrix}$$

$$= d\Theta$$

Also verify $d\Omega = \Omega \wedge \Omega$.

Theorem (II.7.3): $\Omega = (dA)\,{}^tA$

Theorem (II.8.3) (Cartan's structural equations):

(1) $d\Theta = \Omega \wedge \Theta$

(2) $d\Omega = \Omega \wedge \Omega$

Proof of first theorem:

$A = (a_{ij})$ $\quad dA = (da_{ij})$

${}^tA = A^{-1}$ $\quad \Omega =$ matrix of ∇E_i in the basis $\{E_1, E_2, E_3\}$

$dA =$ matrix of ∇E_i in the basis $\{U_1, U_2, U_3\}$

$\Rightarrow \Omega = (dA)A^{-1} = (dA)\,{}^tA$.

Apply to $v = (1,0,0), (0,1,0), (0,0,1)$ to verify.