$$d\left\{\begin{pmatrix} da_{11} & da_{12} & \\ \vdots & & \\ & & \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & \\ \vdots & & \\ & & \end{pmatrix}\right\}$$

(1,2) entry : $(da_{11})a_{21}+(da_{12})a_{22}+(da_{13})a_{23}$

$$= a_{21}\, da_{11} + a_{22}\, da_{12} + a_{23}\, da_{13}$$

differential

$$= da_{21}\wedge da_{11} + da_{22}\wedge da_{12} + da_{23}\wedge da_{13}$$

$$= -\ (1,2) \text{ entry of } dA \wedge {}^{t}(dA)$$

Proof of structural equations:

(Theorem II.8.3).

(1) $$\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = A\, d\xi \qquad d\xi := \begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}$$

$$d\Theta = d(A\, d\xi) = dA \wedge d\xi + A \wedge \underbrace{dd\xi}_{\overset{!}{=}\,0}$$

$$dA = \Omega A$$

$$d\Theta = (\Omega A) \wedge d\xi = \Omega \wedge (A\, d\xi)$$
$$= \Omega \wedge \Theta .$$

(2) $$d\Omega = d\left((dA)({}^tA)\right)$$
$$= d(dA)\, {}^tA - dA \wedge d({}^tA)$$
$$= -dA \wedge {}^t(dA) = -dA \wedge d({}^tA)$$

$$\cancel{\Omega \wedge \Omega = (dA\, {}^tA) \wedge (dA\, {}^tA)}$$
$$\cancel{= (dA\, {}^tA) \wedge {}^t(A\, {}^t(dA))}$$
$$\cancel{= dA \wedge {}^tA\, {}^t(A\, {}^t(dA))}$$
$$\cancel{= dA \wedge {}^t(A\, {}^t(dA)\, A)}$$

$$= -(\underbrace{dA\,{}^tA}\, A)\wedge d({}^tA)$$
$$= -\Omega \wedge {}^t(dA\,{}^tA) = -\Omega\wedge{}^t\Omega$$
$$= +\,\Omega\wedge\Omega$$

$\square$.

<u>Back to surfaces in $\mathbb{R}^3$:</u>

$M \subset \mathbb{R}^3$ a surface.

$\mathcal{U} \subset M$ <u>$\mathcal{U}$ open</u> = image of $\varphi : D \longrightarrow M$.

A frame field $\{E_1, E_2, E_3\}$ is adapted to M (on $\mathcal{U}$) if E_3 is normal to M everywhere on $\mathcal{U}$.

<u>example</u>: Spherical frame field is adapted to the sphere of radius ρ.

The shape operator for M is then

$$S_p(v) = -\nabla_v E_3 .$$
$$= -\left(\omega_{31}(v) E_1 + \omega_{32}(v) E_2\right)$$
$$\boxed{S_p(v) = \omega_{13}(v) E_1 + \omega_{23}(v) E_2}$$

In this case the structural equations become the fundamental equations: We restrict ourselves to M, meaning we only look at vectors tangent to M.

$\theta_1, \theta_2, \theta_3$: 1-forms on M.

θ_i : linear forms on tangent vectors to M.

$\{E_1, E_2\}$ basis of $T_p M$

$\theta_i(E_j) = \delta_{ij}$ $i \neq j$ $i = 1, 2$.

$\theta_3(E_1) = \theta_3(E_2) = 0$.

So $\Theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ 0 \end{pmatrix}$, $\Omega = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix}$

$$m\,T_pM$$

$$d\Theta = \Omega \wedge \Theta = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \wedge \begin{pmatrix} \theta_1 \\ \theta_2 \\ 0 \end{pmatrix}$$

$$\Rightarrow \boxed{\begin{aligned} d\theta_1 &= \omega_{12} \wedge \theta_2 \\ d\theta_2 &= -\omega_{12} \wedge \theta_1 = \omega_{21} \wedge \theta_1 \end{aligned}}$$

$$d\Omega = \Omega \wedge \Omega$$

$$\begin{pmatrix} 0 & d\omega_{12} & d\omega_{13} \\ -d\omega_{12} & 0 & d\omega_{23} \\ -d\omega_{13} & -d\omega_{23} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_{12} & \omega_{13} \\ -\omega_{12} & 0 & \omega_{23} \\ -\omega_{13} & -\omega_{23} & 0 \end{pmatrix} \wedge \begin{pmatrix} \text{itself} \end{pmatrix}$$

$$\boxed{\begin{aligned} d\omega_{12} &= -\omega_{13} \wedge \omega_{23} \\ d\omega_{13} &= \omega_{12} \wedge \omega_{23} \\ d\omega_{23} &= -\omega_{12} \wedge \omega_{13} \end{aligned}}$$

<u>Matrix of the shape operator in the basis</u> $\{E_1, E_2\}$:

$$S_p(E_1) = \omega_{13}(E_1)E_1 + \omega_{23}(E_1)E_2$$

$$S_p(E_2) = \omega_{13}(E_2)E_1 + \omega_{23}(E_2)E_2$$

$$S_p = \begin{pmatrix} \omega_{13}(E_1) & \omega_{13}(E_2) \\ \omega_{23}(E_1) & \omega_{23}(E_2) \end{pmatrix}$$

$$K = \omega_{13}(E_1)\,\omega_{23}(E_2) - \omega_{13}(E_2)\,\omega_{23}(E_1)$$

$$= \omega_{13} \wedge \omega_{23}\,(E_1, E_2)$$

$$= -(d\omega_{12})\,(E_1, E_2)$$

$$\Rightarrow \boxed{\begin{array}{l} \omega_{13} \wedge \omega_{23} = K\, \theta_1 \wedge \theta_2 \\ \text{or} \quad -d\omega_{12} = K\, \theta_1 \wedge \theta_2 \end{array}}$$

$$2H = \omega_{13}(E_1) + \omega_{23}(E_2)$$

$$= \omega_{13}(E_1)\,\theta_2(E_2) + \omega_{23}(E_2)\,\theta_1(E_1)$$

$$= (\omega_{13} \wedge \theta_2)(E_1, E_2) + (\omega_{23} \wedge \theta_1)(E_2, E_1)$$

$$= (\theta_1 \wedge \omega_{23})(E_1, E_2) + (\omega_{13} \wedge \theta_2)(E_1, E_2)$$

So $\boxed{\left(\theta_1 \wedge \omega_{23} + \omega_{13} \wedge \theta_2\right) = 2H\, \theta_1 \wedge \theta_2.}$

Lemma: ω_{12} is the only 1-form on M satisfying

$$d\theta_1 = \omega_{12} \wedge \theta_2$$

$$d\theta_2 = -\omega_{12} \wedge \theta_1$$

Proof: $\{\theta_1, \theta_2\}$ form a basis of 1-forms on M.

So if ω is an arbitrary 1-form $\omega = f\theta_1 + g\theta_2$ where f, g are differentiable functions on M.

In fact $f = \omega(E_1)$, $g = \omega(E_2)$ because $\omega(E_1) = f\theta_1(E_1) + g\theta_2(E_1)$

$$= f$$

Suppose ω is a 1-form s.t.

$$d\theta_1 = \omega \wedge \theta_2$$

and $d\theta_2 = -\omega \wedge \theta_1$

we prove $\omega = \omega_{12}$:
By our observation, we need to show: $\omega(E_1) = \omega_{12}(E_1)$ and
$\omega(E_2) = \omega_{12}(E_2)$.

$$d\theta_1(E_1, E_2) = (\omega \wedge \theta_2)(E_1, E_2)$$
$$= \omega(E_1)\,\theta_2(E_2) = \omega(E_1)$$
$$= (\omega_{12} \wedge \theta_2)(E_1, E_2) = \omega_{12}(E_1)$$

$$d\theta_2(E_1, E_2) = -(\omega \wedge \theta_1)(E_1, E_2)$$
$$= +\omega(E_2)\,\theta_1(E_1) = \omega(E_2)$$
$$= -(\omega_{12} \wedge \theta_2)(E_1, E_2) = \omega_{12}(E_2)$$

$\square$.

So $\omega = \omega_{12}$

Back to an abstract surface M :

$\varphi : D \longrightarrow M$ a coordinate chart.

$p \in M$ $\quad q = \varphi^{-1}(p)$.

$$\mathbb{R}^2 = T_q D \xrightarrow{\cong} T_p M$$
$$(1,0) \longleftrightarrow \varphi_u$$
$$(0,1) \longleftrightarrow \varphi_v$$

Suppose we are given a metric $\langle\, ,\, \rangle$ on M

Define $E_1 := \frac{1}{\|\varphi_u\|} \varphi_u$

$$E_2 := \frac{1}{\left\|\varphi_v - \frac{\langle \varphi_u, \varphi_v \rangle}{\langle \varphi_u, \varphi_u \rangle} \varphi_u\right\|} \left(\varphi_v - \frac{\langle \varphi_u, \varphi_v \rangle}{\langle \varphi_u, \varphi_u \rangle} \varphi_u \right)$$