Practice problems:

(1) V vector field along $\alpha = \alpha(t)$
(defined in a neighborhood of α)

Suppose that we have a well-defined frame field in a neighborhood of α: $\{E_1, E_2\}$.

$\varphi =$ angle between E_1 and V

$\varphi(t)$ change in angle from t_1 to t_2: $\varphi(t_2) - \varphi(t_1)$

$$\varphi(t_2) - \varphi(t_1) = \int_{t_1}^{t_2} \varphi'(t)\,dt.$$

If V has constant length c, then by the lemma: V parallel

$$\Leftrightarrow \quad \varphi'(t) = -\,\omega_{12}(\alpha').$$

So two parallel vector fields have the same φ' and rotate by the same angle.

What happens if we choose a different frame field $\{\overline{E}_1, \overline{E}_2\}$?

$\psi =$ angle between E_1 and $\overline{E}_1$.

<u>then</u>: if the two frame fields have the same orientation:

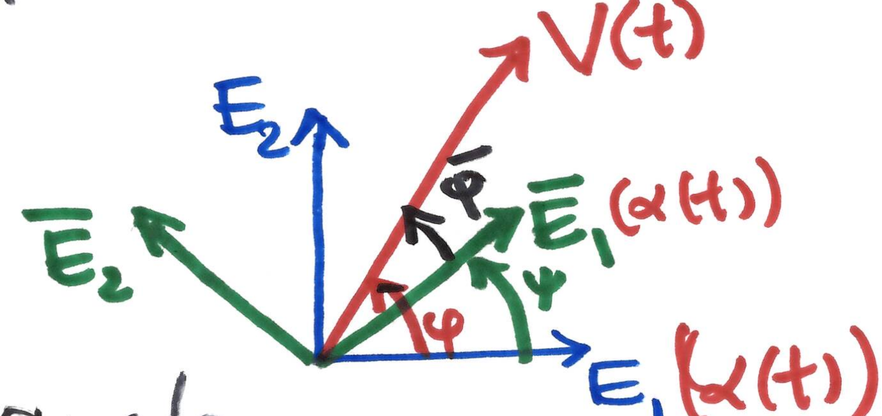
$$\overline{\omega}_{12} = \omega_{12} + d\psi$$

and $\overline{\theta}_1 \wedge \overline{\theta}_2 = \theta_1 \wedge \theta_2$.

if the two frame fields have opposite orientations:

$$\overline{\omega}_{12} = -\omega_{12} - d\psi$$

$$\overline{\theta}_1 \wedge \overline{\theta}_2 = -\theta_1 \wedge \theta_2$$



if $\overline{\varphi} =$ angle between V and $\overline{E}_1$, then

$$\overline{\varphi} = \varphi - \psi$$

So $\overline{\varphi}' = \varphi' - \psi'$

$$\left(-\overline{\omega}_{12}(\alpha') = -\omega_{12}(\alpha') - \psi'\right)$$

<u>or</u>: $\overline{\varphi}(t_2) - \overline{\varphi}(t_1) = \varphi(t_2) - \varphi(t)$

$$- \left(\psi(2) - \psi(t_1)\right)$$

So in general, the angle change <u>does</u> depend on the choice of frame field. However if α is a closed curve and we go around it once, it does not depend on the choice of frame field because $p(t_1) = p(t_2) = \alpha(t_1) = \alpha(t_2)$

and $E_1(p(t_1)) = E_1(p(t_2))$

$$\overline{E}_1(p(t_1)) = \overline{E}_1(p(t_2))$$

and $\psi(t_1) = \psi(t_2)$.

Holonomy is well defined up to multiples of 2π.

<u>Conformal maps and structures:</u>

$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}^4.$

in (5) & (6) : $\varphi_u \cdot \varphi_v = 0$

and $\varphi_u \cdot \varphi_u = \varphi_v \cdot \varphi_v$

So the map $\varphi : \mathbb{R}^2 \to M \subset \mathbb{R}^4$ is conformal.

Can define a frame field on $\mathbb{R}^2$ for the new metric induced from φ : this is the metric on M induced from $\mathbb{R}^4$.

$V \in T_p M \qquad V = a\varphi_u + b\varphi_v$

$$\|V\| = \sqrt{a^2 E + b^2 G}$$

$E = G = \lambda^2 \qquad \lambda > 0$

$$\|V\| = \sqrt{a^2+b^2} \cdot \lambda.$$

$\sqrt{a^2+b^2}$ = Euclidean length of V in $\mathbb{R}^2$.

$\|V\|$ = length of V in metric on M.

$$E_1 := \frac{\varphi_u}{\|\varphi_u\|} = \frac{1}{\lambda}\varphi_u$$

$$E_2 := \frac{\varphi_v}{\|\varphi_v\|} = \frac{1}{\lambda}\varphi_v$$

$$\theta_1 = \lambda\, du \qquad \theta_2 = \lambda\, dv.$$

$$d\theta_1 = \omega_{12}\wedge\theta_2 \quad , \quad d\theta_2 = \omega_{21}\wedge\theta_1$$

$$\omega_{12} = \omega_{12}(E_1)\theta_1 + \omega_{12}(E_2)\theta_2$$

$$\omega_{12}(E_1) = d\theta_1(E_1, E_2)$$

$$\omega_{12}(E_2) = d\theta_2(E_1, E_2)$$

$$d\theta_1 = d\lambda\wedge du = -\frac{\partial\lambda}{\partial v}\, du\wedge dv$$

$$d\theta_2 = d\lambda\wedge dv = \frac{\partial\lambda}{\partial u}\, du\wedge dv$$

$$\Rightarrow \omega_{12} = -\frac{1}{\lambda^2}\frac{\partial\lambda}{\partial v}\theta_1 + \frac{1}{\lambda^2}\frac{\partial\lambda}{\partial u}\theta_2$$

$$\omega_{12} = -\frac{1}{\lambda}\frac{\partial \lambda}{\partial v}\,du + \frac{1}{\lambda}\frac{\partial \lambda}{\partial u}\,dv$$

$$d\omega_{12} = d\left(-\frac{1}{\lambda}\frac{\partial \lambda}{\partial v}\right)\wedge du$$
$$+ d\left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial u}\right)\wedge dv.$$

$$\theta_1 \wedge \theta_2 = \lambda^2\, du \wedge dv$$

$$\frac{\partial}{\partial u}\left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial u}\right) = \frac{\lambda \frac{\partial^2 \lambda}{\partial u^2} - \left(\frac{\partial \lambda}{\partial u}\right)^2}{\lambda^2}.$$

$$d\omega_{12} = \frac{1}{\lambda^2}\left(\left(\frac{\partial \lambda}{\partial v}\right)^2 - \lambda\frac{\partial^2 \lambda}{\partial v^2}\right) dv \wedge du$$
$$+ \frac{1}{\lambda^2}\left(\lambda\frac{\partial^2 \lambda}{\partial u^2} - \left(\frac{\partial \lambda}{\partial u}\right)^2\right) du \wedge dv$$
$$= \frac{1}{\lambda^2}\left[\lambda\frac{\partial^2 \lambda}{\partial u^2} + \lambda\frac{\partial^2 \lambda}{\partial v^2} - \left(\frac{\partial \lambda}{\partial u}\right)^2 - \left(\frac{\partial \lambda}{\partial v}\right)^2\right]$$
$$du \wedge dv$$

$$\Rightarrow K = -\frac{1}{\lambda^4}\left[\lambda\frac{\partial^2 \lambda}{\partial u^2} + \lambda\frac{\partial^2 \lambda}{\partial v^2} - \left(\frac{\partial \lambda}{\partial u}\right)^2 - \left(\frac{\partial \lambda}{\partial v}\right)^2\right]$$

Apply the formulas to exercises (5) and (6):

in (5):

$$\varphi_u = (\sinh u \cos v, \sinh u \sin v, \cosh u \cos v, \cosh u \sin v)$$

$$\varphi_v = (-\cosh u \sin v, \cosh u \cos v, -\sinh u \sin v, \sinh u \cos v)$$

$$E = G = \varphi_u \cdot \varphi_u = \varphi_v \cdot \varphi_v$$

$$= \sinh^2 u + \cosh^2 u = 1 + 2\sinh^2 u$$

$$\lambda = \sqrt{1 + 2\sinh^2 u}$$

$$\frac{\partial \lambda}{\partial v} = 0 \qquad \frac{\partial \lambda}{\partial u} = \frac{2\sinh u \cosh u}{\sqrt{1+2\sinh^2 u}}$$

$$\frac{\partial^2 \lambda}{\partial u^2} = \frac{2\sqrt{1+2\sinh^2 u} \cdot (\sinh^2 u + \cosh^2 u)}{\cancel{\sqrt{\ }}(1+2\sinh^2 u)}$$

$$- \frac{4\sinh u \cosh u \cdot \frac{\sinh u \cosh u}{\sqrt{1+2\sinh^2 u}}}{(1+2\sinh^2 u)}$$

$$= \frac{2\sin^2 hu + 2\cos^2 hu + 4\sin^4 hu}{\left(\sqrt{1+2\sin^2 hu}\right)^3}$$

$$= \frac{2 + 4\sin^2 hu + 4\sin^4 hu}{\left(1+2\sin^2 hu\right)^{3/2}}$$

$\vdots$ compute K.

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(6): $\varphi(u,v) = \left(u, v, uv, \frac{u^2-v^2}{2}\right)$

$\varphi_u = (1, 0, v, u)$

$\varphi_v = (0, 1, u, -v)$

$E = 1 + u^2 + v^2 = G.$

$\lambda = \sqrt{1+u^2+v^2}$

$$\frac{\partial \lambda}{\partial u} = \frac{u}{\sqrt{1+u^2+v^2}}$$

$$\frac{\partial^2 \lambda}{\partial u^2} = \frac{\sqrt{1+u^2+v^2} - u \cdot \frac{u}{\sqrt{1+u^2+v^2}}}{1+u^2+v^2}$$

$$\frac{\partial^2 \lambda}{\partial u^2} = \frac{1+v^2}{(1+u^2+v^2)^{3/2}}$$

(a) $\vdots$ compute K.

(b) <u>area form</u>: $\theta_1 \wedge \theta_2$

$$\theta_1 = \lambda\, du \qquad \theta_2 = \lambda\, dv$$

$$\theta_1 \wedge \theta_2 = \lambda^2\, du \wedge dv$$

$$= (1+u^2+v^2)\, du \wedge dv$$

$$\text{area} = \int_0^1 \int_0^1 (1+u^2+v^2)\, du\, dv$$