$v^n - w^n - dh^n$ factors through d^n if and only if its kernel contains the kernel of d^n:

$$\begin{tikzcd} C^n \arrow[r, "v^n - w^n - dh^n"] \arrow[d, "d^n"'] & I^n \\ C^{n+1} \arrow[ur, dashed, "h^{n+1}"'] & \end{tikzcd}$$

Since $C^\cdot$ is a resolution of A, $\operatorname{Ker} d^n = \operatorname{Im} d^{n-1}$.

So $v^n - w^n - dh^n$ factors through d^n iff its kernel contains $\operatorname{Im} d^{n-1}$ $\iff$ $(v^n - w^n - dh^n)d = 0$

To check this, write $(v^n - w^n - dh^n)d =$

$$(v^n - w^n)d - d(h^n d) = (v^n - w^n)d - d(v^{n-1} - w^{n-1} - dh^{n-1})$$

$$= (v^n - w^n)d - d(v^{n-1} - w^{n-1}) = 0$$

$\square$

Properties of R^iF:

(1) $\forall i$ R^iF is an additive functor from $\mathcal{A}$ to $\mathcal{B}$.

(Can also show that $R^iF(A \oplus B) = R^iF(A) \oplus R^iF(B)$)

(2) There is a natural isomorphism $F \cong R^0F$

(3) If $I \in \mathrm{Ob}(\mathcal{A})$ is injective, then $R^iF(I) = 0$ $\forall i > 0$

(4) For any short exact sequence

$$0 \to A' \to A \to A'' \to 0$$

there are coboundary morphisms $\delta^i : R^iF(A'') \to R^{i+1}F(A')$

s.t.

$$\cdots \to R^iF(A') \to R^iF(A) \to R^iF(A'') \xrightarrow{\delta^i} R^{i+1}F(A') \to \cdots$$

is exact.

Furthermore, the δ^i are functorial, i.e., given a morphism

$$\begin{array}{ccccccccc} 0 & \to & A' & \to & A & \to & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \to & B & \to & B'' & \to & 0 \end{array}$$

the coboundary morphisms form commutative diagrams

$$\begin{array}{ccc} R^iF(A'') & \xrightarrow{\delta^i} & R^{i+1}F(A') \\ \downarrow & \circlearrowleft & \downarrow \\ R^iF(B'') & \xrightarrow{\delta^i} & R^{i+1}F(B') \end{array}$$

Sketch of proof: To construct δ^i, choose injective resolutions $I'^{\bullet}$ and $I''^{\bullet}$ of A' and A'', use Lemma 2 above to construct morphisms fitting in a commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I'^{\bullet} \arrow[r] & I'^{\bullet} \oplus I''^{\bullet} \arrow[r] & I''^{\bullet} \arrow[r] & 0 \end{tikzcd}$$

<u>Definition</u>: An object J is called acyclic for F if

$$R^iF\,(J) = 0 \quad \forall\, i > 0.$$

<u>Remark</u>: One can acyclic resolution instead of injective resolutions to compute the right derived functors.

Given an acyclic resolution $C^{\cdot}$ of A and an injective resolution $I^{\cdot}$ of A, by Lemma 2, $\exists$ a morphism $u^{\cdot} : C^{\cdot} \to I^{\cdot}$ unique up to homotopy extending $Id : A \to A$.

u^i induces morphisms $H^i(F(C^\cdot)) \longrightarrow H^i(F(I^\cdot))$

$$\begin{array}{c} H^i(F(I^\cdot)) \\ \| \\ R^iF(A) \end{array}$$

which one can show are isomorphisms.

Now we move towards defining cohomology of sheaves on schemes.

Lemma 3: The category of modules over a ring R with a unit (not necessarily commutative) has enough injectives.

Idea of proof: The R-module $R^* := \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ is injective, and we can embed cyclic R-modules into R^*.

$\Rightarrow$ Lemma 4: For a ringed space $(X, \mathcal{O}_X)$, the category of sheaves of $\mathcal{O}_X$-modules has enough injectives.

<u>Idea of proof</u>: Given an $\mathcal{O}_X$-module $\mathcal{F}$, embed each stalk $\mathcal{F}_x$ $(x \in X)$ into an injective $\mathcal{O}_{X,x}$-module I_x. Then put $\mathcal{I} := \prod_{x \in X} (i_x)_* I_x$

$i_x : \{x\} \hookrightarrow X$

$\forall$ $\mathcal{O}_X$-module $\mathcal{G}$, we have $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I})$

$= \prod_{x \in X} \mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, (i_x)_* I_x)$,

and $\forall$ $x \in X$, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{G}, (i_x)_* I_x) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$

$\Longrightarrow$ <u>Corollary</u>: Apply this to $(X, \mathbb{Z}_X)$ where X is any topological space and $\mathbb{Z}_X$ is the locally constant sheaf.

We obtain that the category of sheaves of abelian groups on X has enough injectives.

<u>Fact</u>: The category of modules over a ring R has enough projectives: any module is the quotient of a free module which is projective.

However, the category of sheaves of abelian groups on a topological space X does <u>not</u> necessarily have enough projectives.

<u>Example</u>: Suppose X is a topological space s.t.

$\exists\ \underset{\text{closed point}}{x \in X}$ s.t. $\forall$ open $V \ni x$, $\exists$ an open $U \ni x$

s.t. $U \subsetneq V$ and U is connected.

For such U, V, let $\mathbb{Z}_{X,U} := i_! \mathbb{Z}_U$ be the extension by 0 of the constant sheaf $\mathbb{Z}_U$, i.e.,

$$\mathbb{Z}_{X,U}(W) = \mathbb{Z} \text{ if } W \subset U, \text{ and } \mathbb{Z}_{X,U}(W) = 0, \text{ if } W \not\subset U.$$

Let $\mathbb{Z}_x$ be the skyscraper sheaf supported at x with stalk $\mathbb{Z}$, i.e., $\mathbb{Z}_x(W) = \mathbb{Z}$, if $x \in W$, and $\mathbb{Z}_x(W) = 0$ if $x \notin W$.

Note that there is a surjective morphism

$$\mathbb{Z}_{X,U} \twoheadrightarrow \mathbb{Z}_x$$

<u>Claim</u>: $\mathbb{Z}_x$ is NOT the quotient of a projective sheaf.

Otherwise if $\mathcal{P} \twoheadrightarrow \mathbb{Z}_x$ with $\mathcal{P}$ projective, then

$\exists$ lift

$$\begin{tikzcd} \mathcal{P} \arrow[r] \arrow[rr, bend right] & \mathbb{Z}_{X,U} \arrow[r] & \mathbb{Z}_x \end{tikzcd}$$

We have $\mathbb{Z}_{X,U}(V) = 0$ because $V \not\subset U$

$\Rightarrow$ the map $\mathcal{F}(V) \longrightarrow \mathbb{Z}_x(V)$ which factors through $\mathbb{Z}_{X,U}(V)$ is 0.

If we fix x and $\mathcal{F}$, we can change U and V as above. This means that $\forall\ V \ni x$ open $\mathcal{F}(V) \to \mathbb{Z}_x(V)$ is 0. $\Rightarrow$ the map on stalks $\mathcal{F}_x \longrightarrow \mathbb{Z}_x$ is 0: contradiction. $\square$