

Definition 1: For a topological space  $X$ , the cohomology functors  $H^i(X, \cdot) : \mathcal{A}b(X) \rightarrow \mathcal{A}b$  are the right-derived functors of the global sections functor  $\Gamma(\cdot) : \mathcal{A}b(X) \rightarrow \mathcal{A}b$ , i.e.,  $H^i(X, \mathcal{F}) = R^i \Gamma(\mathcal{F})$ .  
In particular,  $H^0(X, \mathcal{F}) = \Gamma(\mathcal{F})$ .

Notation: From now on, we will use  $H^0(X, \mathcal{F})$  to denote the global sections of  $\mathcal{F}$ .

Definition 2: A sheaf  $\mathcal{F}$  on a topological space  $X$  is called flasque if,  $\forall U \subset X$  open, the restriction map  $H^0(X, \mathcal{F}) \rightarrow H^0(U, \mathcal{F})$  is surjective.

Lemma 1: On a ringed space  $(X, \mathcal{O}_X)$ , injective sheaves of  $\mathcal{O}_X$ -modules are flasque.

Proof: Suppose  $\mathcal{F}$  is an injective sheaf of  $\mathcal{O}_X$ -modules.

For any open set  $V \subset X$ , we want to show

$$H^0(X, \mathcal{F}) \longrightarrow H^0(V, \mathcal{F}).$$

$$\begin{array}{ccc} \parallel & & \parallel \\ \text{Hom}(\mathcal{O}_X, \mathcal{F}) & & \text{Hom}(\mathcal{O}_U, \mathcal{F}|_U) \end{array}$$

$$i: U \hookrightarrow X \qquad \text{Hom}(i_! \mathcal{O}_U, \mathcal{F})$$

where  $i_! \mathcal{O}_U$  is the extension of  $\mathcal{O}_U$  to  $X$  by 0:

$$\begin{aligned} i_! \mathcal{O}_U(V) &= \mathcal{O}_X(V) = \mathcal{O}_U(V) \text{ if } V \subset U \\ &= 0 \text{ otherwise.} \end{aligned}$$

$$j: X \setminus U \hookrightarrow X$$

Now (see Ex II.1.19) we have  $0 \rightarrow i_! \mathcal{O}_U \rightarrow \mathcal{O}_X \rightarrow j_* (\mathcal{O}_X|_{X \setminus U}) \rightarrow 0$

$\text{Hom}(\cdot, \mathcal{F})$  is exact  $\Rightarrow \text{Hom}(\mathcal{O}_X, \mathcal{F}) \rightarrow \text{Hom}(i_! \mathcal{O}_U, \mathcal{F}) \quad \square$

Lemma 2: Flasque sheaves are acyclic for the global sections functor.

Proof: Let  $\mathcal{F}$  be a flasque sheaf on the topological space  $X$ .  $\exists$  an injective sheaf  $\mathcal{J}$  of abelian groups on  $X$  s.t.  $\mathcal{F} \hookrightarrow \mathcal{J}$ . Let  $\mathcal{G}$  be the quotient:

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{J} \rightarrow \mathcal{G} \rightarrow 0.$$

Ex: II.1.16: (and Lemma 1)  $\mathcal{F}$  &  $\mathcal{J}$  are flasque  $\Rightarrow \mathcal{G}$  flasque

$\mathcal{J}$  injective  $\Rightarrow \mathcal{J}$  acyclic.  $\Rightarrow H^i(X, \mathcal{J}) = 0 \quad \forall i > 0$

$\mathcal{F}$  flasque (Ex. II.1.16)  $\Rightarrow$  the sequence

$$0 \rightarrow H^0(X, \mathcal{F}) \rightarrow H^0(X, \mathcal{J}) \rightarrow H^0(X, \mathcal{G}) \rightarrow 0$$

is exact.

We have the long exact sequence of cohomology:

$$\begin{array}{ccccccccccc}
 0 \rightarrow & H^0(\mathcal{F}) & \rightarrow & H^0(\mathcal{G}) & \rightarrow & H^0(\mathcal{Y}) & \rightarrow & H^1(\mathcal{F}) & \rightarrow & H^1(\mathcal{G}) & \rightarrow & H^1(\mathcal{Y}) \\
 & & & & & & & & & & & \parallel \\
 & & & & & & & & & & & 0 \\
 & & & & & & & & & & & \dots \\
 & & & & & & & & & & & \parallel \\
 & & & & & & & & & & & 0 \\
 & & & & & & & & & & & \dots
 \end{array}$$

$$\Rightarrow H^1(\mathcal{F}) = 0$$

$$\text{and } H^i(\mathcal{Y}) \cong H^{i+1}(\mathcal{F}) \quad \forall i \geq 1$$

by induction  $\Rightarrow H^i(\mathcal{F}) = 0 \quad \forall \mathcal{F}$  flasque and  $i > 0$ .  
 $\square$

Corollary: All cohomology groups on a ringed space  $(X, \mathcal{O}_X)$  are  $H^0(X, \mathcal{O}_X)$ -modules.

Proof: Flasque sheaves are acyclic by Lemma 2. So we can use flasque resolutions to compute  $H^i$ . On any  $(X, \mathcal{O}_X)$ ,

the injective sheaves of  $\mathcal{O}_X$ -modules are flasque,  
so we can use resolutions by injective sheaves of  $\mathcal{O}_X$ -mod.  
to compute  $H^i$ .  $\Rightarrow H^i(X, \mathcal{F})$  are  $H^0(X, \mathcal{O}_X)$ -modules  
 $\forall \mathcal{F} \mathcal{O}_X$ -mod.  $\square$ .

Theorem: Let  $X$  be a noetherian topological space  
of dimension  $n$ . Then, for any sheaf  $\mathcal{F}$  of abelian  
groups on  $X$  and any  $i > n$ ,  $H^i(X, \mathcal{F}) = 0$

Proof: Step 1: Reduce to the case where  $X$  is irreducible  
by induction on the number of irreducible components:  
If  $Y \subset X$  is an irreducible component, put  
 $U := X \setminus Y$ ,  $\bar{U}$  has one irreducible component  
less than  $X$

(Ex. II.1.19), we have the exact sequence

$$0 \rightarrow i_! \mathcal{F}|_U \rightarrow \mathcal{F} \rightarrow j_* j^{-1} \mathcal{F} \rightarrow 0$$

where  $i: U \hookrightarrow X$   $j: Y \hookrightarrow X$

So the vanishing of  $H^k(i_! \mathcal{F}|_U)$  and  $H^k(j_* j^{-1} \mathcal{F})$

imply the vanishing of  $H^k(\mathcal{F})$ .

$$(1) H^k(i_! \mathcal{F}|_U) = H^k((i_! \mathcal{F}|_U)|_{\bar{U}}) \quad \text{on } \bar{U}$$

$$(2) H^k(j_* j^{-1} \mathcal{F}) = H^k(j^{-1} \mathcal{F}) \quad \text{on } Y.$$

For (2) we have a Lemma:

Lemma 3:  $\forall$  closed subset  $Y \xrightarrow{j} X$ , if  $\mathcal{F}$  is a sheaf of abelian groups on  $Y$ , then  $H^k(Y, \mathcal{F}) = H^k(X, j_* \mathcal{F})$ .

Proof: If  $\mathcal{J}$  is a flasque resolution of  $\mathcal{F}$  on  $Y$ ,  
then  $j_* \mathcal{J}^*$  is a flasque resolution of  $j_* \mathcal{F}$  on  $X$ .

$$\text{and } \forall l \quad H^0(Y \cap U, \mathcal{J}^l) = H^0(U, j_* \mathcal{J}^l)$$

$$U \subset X \text{ open} \quad H^0(Y, \mathcal{J}^l) = H^0(X, j_* \mathcal{J}^l)$$

$$\Rightarrow H^k(Y, \mathcal{F}^l) = H^k(X, j_* \mathcal{F}^l) \quad \square.$$

For (1), using Ex. II.1.19, the stalk  $(i_! \mathcal{F}^l|_U)_x = 0 \quad \forall x \notin U$

this implies that the natural map  $i_! \mathcal{F}^l|_U \rightarrow \mu_* \mu^{-1}(i_! \mathcal{F}^l|_U)$   
where  $\mu: \bar{U} \hookrightarrow X$  is an isomorphism.

Hence, by Lemma 3,  $H^l(i_! \mathcal{F}^l|_U) = H^l((i_! \mathcal{F}^l|_U)_{\bar{U}}) \quad \forall l$ .

Step 2: True for  $X = \text{point} = \{x\}$ .

In this case we have an equivalence of categories

$$\mathcal{A}b(X) \xrightarrow{H^0 = \text{stalk at } x} \mathcal{A}b$$

The global sections functor is exact in this case.

$$\Rightarrow H^i = 0 \quad \forall i > 0.$$

Step 3: Induction on dimension, and reduction to the case of a finitely generated sheaf.

Put  $B := \bigcup_{U \subset X} \mathcal{F}|_U$  Given a subset  $\alpha$  of  $B$ ,

we let  $\mathcal{F}_\alpha \subset \mathcal{F}$  be the smallest subsheaf of  $\mathcal{F}$  containing  $\alpha$ : this is the subsheaf of  $\mathcal{F}$  generated by  $\alpha$ . If  $\mathcal{F}_\alpha$  can be generated by a finite



subset of  $B$ , we say it is finitely generated.

The set  $A$  of all finitely generated subsheaves of  $\mathcal{F}$  is a directed set, and

$$\mathcal{F} = \varinjlim_{\alpha \in A} \mathcal{F}_\alpha$$

$\forall \alpha \in A$  we have  $\mathcal{F}_\alpha \hookrightarrow \mathcal{F} = \varinjlim_{\alpha} \mathcal{F}_\alpha$

$$\Rightarrow H^k(\mathcal{F}_\alpha) \longrightarrow H^k(\mathcal{F}) = H^k(\varinjlim_{\alpha} \mathcal{F}_\alpha)$$

$$\Rightarrow \varinjlim_{\alpha} H^k(\mathcal{F}_\alpha) \longrightarrow H^k(\mathcal{F}) = H^k(\varinjlim_{\alpha} \mathcal{F}_\alpha)$$

Ex. II.1.11 this map is an isom. when  $i=0$ .

One can show this implies the map is an isom  $\forall i$ .

$\Rightarrow$  Enough to show the theorem for  
finitely generated sheaves.