

a scheme X is affine iff $\exists f_1, \dots, f_n \in H^0(X, \mathcal{O}_X)$

generating the unit ideal s.t. $\forall i$

$X_{f_i} := \{x \in X \mid f_i \notin \mathfrak{m}_x\}$ is affine.

Let $P \in X$ be a closed point, U an affine neighborhood of P and put $Y := X \setminus U$. Let \mathcal{I}_P be the skyscraper sheaf with support $\{P\}$ and stalk $k = k(P)$.

We have the exact sequence.

$$0 \longrightarrow \mathcal{I}_{Y \amalg \{P\}} \longrightarrow \mathcal{I}_Y \longrightarrow k_P \longrightarrow 0$$

$\mathcal{I}_{Y \amalg \{P\}} = \mathcal{I}_Y \oplus \mathcal{I}_P$
 \uparrow sheaf of ideals of Y for some scheme structure on Y .

The map: $\mathcal{I}_Y \longrightarrow k_P$:

If $V \not\supseteq P$, then $k_P(V) = 0$ by def.

and $\mathcal{I}_Y(V) \longrightarrow k_P(V)$ is zero.

If $V \supseteq P$, then $k_P(V) = k$ and $U \cap V \ni P$

The map is $\mathcal{I}_Y(V) \longrightarrow \mathcal{I}_Y(U \cap V) = \mathcal{O}_X(U \cap V)$

$\longrightarrow \mathcal{O}_{X,P} \longrightarrow k_P = \mathcal{O}_{X,P}/\mathfrak{m}_P = k_P(V)$

The kernel is $\mathcal{I}_{Y \cup \{P\}}$.

The long exact sequence of cohomology:

$$0 \longrightarrow H^0(\mathcal{I}_{Y \cup \{P\}}) \longrightarrow H^0(\mathcal{I}_Y) \longrightarrow H^0(k_P) \longrightarrow H^1(\mathcal{I}_{Y \cup \{P\}})$$

\parallel \parallel

k_P 0 by (c)

$$\Rightarrow H^0(\mathcal{I}_Y) \twoheadrightarrow k_P$$

In particular, $\exists f \in H^0(\mathcal{D}_Y)$ mapping to $1 \in k_P$. This implies $f_P \notin \mathfrak{m}_P$

$$\Rightarrow P \in X_f := \{x \in X \mid f_x \notin \mathfrak{m}_x\}$$

We also have $X_f = U_f := \{x \in U \mid f_x \notin \mathfrak{m}_x\}$
because $Y = X \setminus U$ and $f \in \mathcal{D}_Y \Rightarrow f_x \in \mathfrak{m}_x \forall x \in Y$
 $\Rightarrow Y \cap X_f = \emptyset$

$$\Rightarrow X_f \subset U.$$

$$\Rightarrow X_f = U_f.$$

Since U is affine, so is $U_f = X_f$.

Since X is noetherian, we can cover X with a finite #, say X_{f_1}, \dots, X_{f_n} of such affine open sets.

It remains to prove that f_1, \dots, f_n generate the unit ideal in $H^0(X, \mathcal{O}_X)$.

This means that the map

$$\begin{aligned} H^0(X, \mathcal{O}_X)^{\oplus n} &\longrightarrow H^0(X, \mathcal{O}_X) \\ (a_1, \dots, a_n) &\longmapsto \sum_{i=1}^n a_i f_i \end{aligned}$$

is surjective

This is the map on global sections of the morphism of sheaves:

$$\begin{aligned} \mathcal{O}_X^{\oplus n} &\longrightarrow \mathcal{O}_X \\ (b_1, \dots, b_n) &\longmapsto \sum_{i=1}^n b_i f_i \quad b_i \in \mathcal{O}_X(U) \end{aligned}$$

on any U

This is a surjective morphism of sheaves because

$$X = \bigcup_{i=1}^n X_{f_i} \implies \forall x \in X, \exists i \text{ s.t. } (f_i)_x \notin \mathfrak{m}_x$$

i.e., $(f_i)_n$ generates $\mathcal{O}_{X,n}$.

Let \mathcal{F} be the kernel of this morphism, so that we have the exact sequence:

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{O}_X^{\oplus n} \longrightarrow \mathcal{O}_X \longrightarrow 0.$$

Using the long exact sequence of cohomology, if $H^1(X, \mathcal{F}) = 0$, then $H^0(\mathcal{O}_X^{\oplus n}) \rightarrow H^0(\mathcal{O}_X)$ is surjective.

$$\begin{array}{ccccccc} \text{Filter: } \mathcal{O}_X & \subset & \mathcal{O}_X^{\oplus 2} & \subset & \mathcal{O}_X^{\oplus 3} & \subset & \dots \subset \mathcal{O}_X^{\oplus n} \\ & & \cup & & \cup & & \cup \\ & & \mathcal{F} \cap \mathcal{O}_X^{\oplus 2} & \subset & \mathcal{F} \cap \mathcal{O}_X^{\oplus 3} & \subset & \dots \subset \mathcal{F} \end{array}$$

$$\begin{array}{ccccccc} \text{Filtr: } 0 & \rightarrow & \mathcal{F} \cap \mathcal{O}_X & \rightarrow & \mathcal{F} \cap \mathcal{O}_X^{\oplus 2} & \rightarrow & \mathcal{F} \cap \mathcal{O}_X^{\oplus n} / \mathcal{F} \cap \mathcal{O}_X \rightarrow 0 \\ & & \cap & & \cap & & \cap \\ 0 & \rightarrow & \mathcal{O}_X & \rightarrow & \mathcal{O}_X^{\oplus 2} & \rightarrow & \mathcal{O}_X \rightarrow 0 \end{array}$$

The long exact sequence of cohomology and assumption (c) $\Rightarrow H^i(\mathcal{F} \cap \mathcal{O}_X^{\oplus 2}) = 0 \quad \forall i > 0.$

continue with

$$0 \rightarrow \mathcal{F} \cap \mathcal{O}_X^{\oplus 2} \rightarrow \mathcal{F} \cap \mathcal{O}_X^{\oplus 3} \rightarrow \mathcal{F} \cap \mathcal{O}_X^{\oplus 3} / \mathcal{F} \cap \mathcal{O}_X^{\oplus 2} \rightarrow 0$$

$$0 \rightarrow \mathcal{O}_X^{\oplus 2} \rightarrow \mathcal{O}_X^{\oplus 3} \rightarrow \mathcal{O}_X \rightarrow 0$$

$$\Rightarrow H^i(\mathcal{F} \cap \mathcal{O}_X^{\oplus 3}) = 0 \quad \forall i > 0$$

continue to get $H^i(\mathcal{F}) = 0 \quad \forall i > 0$

$$\Rightarrow H^0(\mathcal{O}_X^{\oplus 2}) \twoheadrightarrow H^0(\mathcal{O}_X)$$

$\Rightarrow f_1, \dots, f_2$ generate the unit ideal.

□.

Cech cohomology: sometimes a practical way
of computing coherent cohomology

The general set-up:

X a topological space

\mathcal{F} a sheaf of abelian groups on X .

$\mathcal{U} = \{U_i \mid i \in I\}$ an open covering of X .

The axiom of choice implies that any set has at least one well-ordering (a total ordering s.t. every non-empty subset has a minimal element).

Choose a well-ordering on I .

For any finite subset $\{i_0 < i_1 < \dots < i_n\} \subset I$,

let U_{i_0, \dots, i_n} be the intersection $U_{i_0} \cap \dots \cap U_{i_n}$.

with $\mu_{i_0, \dots, i_n} : U_{i_0, \dots, i_n} \hookrightarrow X$

and put $\mathcal{F}_{i_0, \dots, i_n} := (\mu_{i_0, \dots, i_n})_* \left(\mathcal{F} |_{U_{i_0, \dots, i_n}} \right)$.

Define coboundary maps:

$$d^n : \prod_{\substack{i_0 < \dots < i_n \in I \\ \in \mathcal{U}}} \mathcal{F}_{i_0, \dots, i_n} \longrightarrow \prod_{\substack{i_0 < \dots < i_{n+1} \\ \in \mathcal{U}}} \mathcal{F}_{i_0, \dots, i_{n+1}}$$

$$\alpha = (\alpha_{i_0, \dots, i_n}) \longmapsto d\alpha$$

$$(d\alpha)_{i_0, \dots, i_{n+1}} = \sum_{p=0}^{n+1} (-1)^p \alpha_{i_0, \dots, \hat{i}_p, \dots, i_{n+1}} \Big|_{U_{i_0, \dots, i_{n+1}}}$$

The coboundaries form the long exact sequence of sheaves:

$$0 \rightarrow \mathcal{F} \rightarrow \prod_{i_0 \in I} \mathcal{F}_{i_0} \xrightarrow{d^0} \prod_{i_0 < i_1 \in \bar{I}} \mathcal{F}_{i_0 i_1} \xrightarrow{d^1} \prod_{i_0 < i_1 < i_2 \in \bar{I}} \mathcal{F}_{i_0 i_1 i_2} \xrightarrow{d^2} \dots$$

The exactness is left as an exercise.

For instance, the fact that the kernel of d^0 is the image of \mathcal{F} is equivalent to the sheaf axioms for \mathcal{F} .

Definition: The Čech cohomology $\check{H}^i(\mathcal{U}, \mathcal{F})$ of \mathcal{F} for the open covering \mathcal{U} is the cohomology of the complex of global sections of the exact sequence of sheaves above. In general, this would depend

on the choice of the well-ordering of I .

We shall prove that for a noetherian separated scheme with an affine cover, the Čech cohomology groups of quasi-coherent sheaves are naturally isomorphic to the usual derived cohomology groups.

More notation: $C^k(\mathcal{F}) := \prod_{i_0 < \dots < i_r \in I} \mathcal{F}_{i_0 \dots i_r}$.

$$C^k(\mathcal{F}) := \prod_{i_0 < \dots < i_r \in I} \mathcal{F}(U_{i_0 \dots i_r})$$

the global sections of $C^k(\mathcal{F})$

Terminology: These are called the Čech k -cochains.

$$H^r(\mathcal{U}, \mathcal{F}) = H^r(C^*(\mathcal{F})) \text{ by definition.}$$

We need some preliminary results, interesting in their own right:

Lemma: X top. space. If \mathcal{F} is flasque, then so are the sheaves $\mathcal{C}^b(\mathcal{F})$.

Proof: The restriction of a flasque sheaf is flasque, direct images of flasque sheaves are flasque, products of flasque sheaves are flasque. \square .

Lemma: If \mathcal{F} is flasque, then $\forall \mathcal{U}$ and all $n > 0$, we have $H^n(\mathcal{U}, \mathcal{F}) = 0$.

Proof: We have the Čech resolution,

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0(\mathcal{F}) \xrightarrow{d^0} \mathcal{C}^1(\mathcal{F}) \rightarrow \dots$$

unwind into short exact sequences.

$$0 \rightarrow \mathcal{F} \rightarrow \mathcal{C}^0 \xrightarrow{d^0} \mathcal{B}^1 \rightarrow 0 \Rightarrow \mathcal{B}^1 \text{ flasque}$$

$$0 \rightarrow \mathcal{B}^1 \xrightarrow{d^1} \mathcal{C}^1 \xrightarrow{d^2} \mathcal{B}^2 \rightarrow 0 \Rightarrow \mathcal{B}^2 \text{ flasque}$$

$$0 \rightarrow \mathcal{B}^l \xrightarrow{\vdots} \mathcal{C}^l \xrightarrow{\vdots} \mathcal{B}^{l+1} \rightarrow 0 \Rightarrow \mathcal{B}^{l+1} \text{ flasque.}$$

\Rightarrow all sequences of global sections obtained from the above are exact.

\Rightarrow the Čech complex of global sections is exact

$$\Rightarrow H^p(\mathcal{U}, \mathcal{F}) = 0 \quad \forall p > 0 \quad \square.$$

Construction: X top. space.

Recall that for any resolution $\mathcal{F} \rightarrow \mathcal{R}^\bullet$ and any injective resolution $\mathcal{F} \rightarrow \mathcal{I}^\bullet$, \exists a morphism

$u^\bullet: \mathcal{R}^\bullet \rightarrow \mathcal{I}^\bullet$ extending $\text{Id}: \mathcal{F} \rightarrow \mathcal{F}$, i.e.,

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{\text{Id}}, & \mathcal{F} \\ \downarrow & \mathcal{D} & \downarrow \\ \mathcal{R}^\bullet & \xrightarrow{u^\bullet} & \mathcal{I}^\bullet \end{array}$$

u^\bullet unique up to homotopy.

Apply this to $\mathcal{F} \rightarrow \mathcal{C}^\bullet(\mathcal{F})$.

$$u^\bullet: \mathcal{C}^\bullet(\mathcal{F}) \longrightarrow \mathcal{I}^\bullet \text{ unique up to homotopy}$$

passing to the complexes of global sections, defines

$$\kappa: \check{H}^k(\mathcal{U}, \mathcal{F}) \longrightarrow H^k(X, \mathcal{F}) \text{ unique.}$$

Theorem: X noetherian scheme, \mathcal{F} quasi-coherent sheaf of \mathcal{O}_X -modules. \mathcal{U} a covering of X by open affine schemes $U_i, i \in I$ s.t. $\forall i_0 < \dots < i_n \in I$, $U_{i_0 \dots i_n}$ is affine. Then the canonical morphism

$$\kappa: \check{H}^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F})$$

is an isomorphism $\forall n$.

Proof: Recall that \mathcal{F} can be embedded in a flasque quasi-coherent sheaf, say \mathcal{G} .