

Theorem: X noetherian scheme, \mathcal{F} quasi-coherent sheaf of \mathcal{O}_X -modules. \mathcal{U} a covering of X by open affine schemes $U_i, i \in I$ s.t. $\forall i_0 < \dots < i_n \in I$, $U_{i_0 \dots i_n}$ is affine. Then the canonical morphism

$$\kappa: \check{H}^n(\mathcal{U}, \mathcal{F}) \longrightarrow H^n(X, \mathcal{F})$$

is an isomorphism $\forall n$.

Proof: Recall that \mathcal{F} can be embedded in a flasque quasi-coherent sheaf, say \mathcal{G} . Let \mathcal{R} be the quotient of \mathcal{G} by \mathcal{F} , so that we have the exact sequence of quasi-coherent sheaves

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow \mathcal{R} \longrightarrow 0$$

This gives us the exact sequences, $\forall k \geq 0$

$$0 \longrightarrow \mathcal{E}^k(\mathcal{F}) \longrightarrow \mathcal{E}^k(\mathcal{y}) \longrightarrow \mathcal{E}^k(\mathcal{R}) \longrightarrow 0$$

Since the \cup_{i_0, \dots, i_p} are affine, we also have the exact sequences of global sections, $\forall k \geq 0$

$$0 \longrightarrow C^k(\mathcal{F}) \longrightarrow C^k(\mathcal{y}) \longrightarrow C^k(\mathcal{R}) \longrightarrow 0.$$

So we also have the short exact sequence of (exact)

complexes:
$$0 \longrightarrow C^{\cdot}(\mathcal{F}) \longrightarrow C^{\cdot}(\mathcal{y}) \longrightarrow C^{\cdot}(\mathcal{R}) \longrightarrow 0$$

which gives the long exact sequence of cohomology

groups
$$0 \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, \mathcal{y}) \longrightarrow \dots$$

The canonical morphisms $\kappa: \check{H}^k(\mathcal{U}, *) \longrightarrow H^k(X, *)$

commute with the morphisms of long exact sequences of cohomology.

So we have commutative diagrams with exact rows:

$$\begin{array}{ccccccc}
 0 \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{F}) & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{G}) & \rightarrow & \check{H}^0(\mathcal{U}, \mathcal{R}) & \rightarrow & \check{H}^1(\mathcal{U}, \mathcal{F}) & \rightarrow & 0 \\
 & \cong \downarrow \kappa & & \cong \downarrow \kappa & & \cong \downarrow \kappa & & \downarrow \kappa & & \\
 0 \rightarrow & H^0(X, \mathcal{F}) & \rightarrow & H^0(X, \mathcal{G}) & \rightarrow & H^0(X, \mathcal{R}) & \rightarrow & H^1(X, \mathcal{F}) & \rightarrow & 0
 \end{array}$$

(note $\check{H}^i(\mathcal{U}, \mathcal{G}) = H^i(X, \mathcal{G}) = 0 \quad \forall i > 0$ because \mathcal{G} is flasque)

$$\Rightarrow \kappa: \check{H}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} H^1(X, \mathcal{F})$$

For $i \geq 1$, we have commutative diagrams:

$$\begin{array}{ccccccc}
 0 \rightarrow & \check{H}^i(\mathcal{U}, \mathcal{R}) & \xrightarrow{\delta} & \check{H}^{i+1}(\mathcal{U}, \mathcal{F}) & \rightarrow & 0 \\
 & \downarrow \kappa & & \downarrow \kappa & & \\
 0 \rightarrow & H^i(X, \mathcal{R}) & \rightarrow & H^i(X, \mathcal{F}) & \rightarrow & 0
 \end{array}$$

We know $\kappa: \check{H}^i(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} H^i(X, \mathcal{F}) \quad \forall$
quasi-coherent sheaves on X . By induction, we
obtain $\kappa: \check{H}^i(\mathcal{U}, \mathcal{F}) \xrightarrow{\cong} H^i(X, \mathcal{F}) \quad \forall i$
and all quasi-coherent sheaves. \square .

Benchmark: When X is a separated Noetherian scheme,
it satisfies the hypothesis of the theorem for any
open affine covering by Ex. II.4.3 from last quarter's homework.

Corollary: If X has a covering by $r+1$ open affine
sets s.t. all the intersections are affine, then
 \forall quasi-coherent sheaves \mathcal{F} and all $i > r$,
 $H^i(X, \mathcal{F}) = 0$.

We use the previous to compute cohomology on projective space:

Let A be a noetherian ring.

$$S := A[X_0, \dots, X_n]$$

$$X := \mathbb{P}_A^n = \text{Proj } S$$

$\forall \mathcal{F}$ sheaf of \mathcal{O}_X -modules: $\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} H^0(X, \mathcal{F}(n))$

Recall $\Gamma_*(\mathcal{O}_X) = S$ as graded S -modules.

and \exists natural map

$$\widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}$$

which is an isomorphism if \mathcal{F} is quasi-coherent.

Theorem: (1) $\forall n \in \mathbb{Z}, \forall i \neq 0, n, H^i(X, \mathcal{O}_X(n)) = 0$
(2) $H^n(X, \mathcal{O}_X(-n-1)) \cong A$
(3) $\forall n \in \mathbb{Z}$, the natural map (cup-product)

$$H^0(X, \mathcal{O}_X(n)) \times H^n(X, \mathcal{O}_X(-n-1)) \rightarrow H^n(X, \mathcal{O}_X(-n-1)) = A$$

is a perfect pairing of finitely generated free A -modules.

Proof: Put $\mathcal{F} := \bigoplus \mathcal{O}_X(n)$. Then \mathcal{F} is quasi-coherent, in fact, locally free of infinite rank. \mathcal{F} has a \mathbb{Z} -grading.

We compute $H^i(\mathcal{F})$, keeping track of the \mathbb{Z} -grading.

(cohomology commutes with arbitrary direct sums: they are direct limits of finite direct sums).

Recall that all cohomology groups are

$$H^0(X, \mathcal{O}_X) = A \text{ - modules.}$$

We compute $\check{H}^i(\mathcal{U}, \mathcal{F})$ for $\mathcal{U} = \{U_0, \dots, U_n\}$ the usual open cover of X .

Recall $U_i = D_+(x_i) \subset X$

$$U_{i_0, \dots, i_r} = D_+(x_{i_0} x_{i_1} \dots x_{i_r}) \subset X$$

$\forall \{i_0, \dots, i_r\}$, we have

$$\mathcal{F}(U_{i_0, \dots, i_r}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{O}_X(n)(U_{i_0, \dots, i_r})$$

$$= S[x_{i_0} \dots x_{i_r}]^{-1}$$

because $\mathcal{O}_X(n)(U_{i_0, \dots, i_r}) = S[x_{i_0} \dots x_{i_r}]^{-1}_{\text{degree } n}$

So we have the Čech complex of \mathcal{F} :

$$0 \rightarrow \prod_{i=0}^n \mathcal{F}(U_i) \rightarrow \prod_{0 \leq i < j \leq n} \mathcal{F}(U_{ij}) \rightarrow \dots \rightarrow \mathcal{F}(U_{0 \dots n}) \rightarrow 0$$

\parallel \parallel \parallel

$$0 \rightarrow \prod_{i=0}^n S[x_i^{-1}] \rightarrow \prod_{0 \leq i < j \leq n} S[x_i^{-1} x_j^{-1}] \rightarrow \dots \rightarrow S[x_0^{-1} \dots x_n^{-1}] \rightarrow 0$$

The kernel of the first map is $H^0(X, \mathcal{F}) = S$ (already known). $H^n(X, \mathcal{F})$ is the cokernel of the last map:

$$\prod_{i=0}^n S[x_0^{-1} \dots \widehat{x_i^{-1}} \dots x_n^{-1}] \rightarrow S[x_0^{-1} \dots x_n^{-1}]$$

The ring $S[x_0^{-1} \dots x_n^{-1}]$ is the free A -module with basis all the Laurent monomials, i.e., monomials with both

positive and negative powers.

The cokernel of the map is the free A -module with basis the monomials where all the X_i have negative power: $\{X_0^{l_0} \cdots X_n^{l_n}, l_i < 0 \forall i\}$

The grading on \mathcal{F} or $S[X_0^{-1} \cdots X_n^{-1}]$ is given by $\sum_{i=0}^n l_i$. So $H^n(X, \mathcal{O}_X(-n)) = 0$ if $n \leq n$

and $H^1(X, \mathcal{O}_X(-n-1-n))$ is isomorphic to the

free A -module with basis $X_0^{l_0} \cdots X_n^{l_n}$ $l_i < 0$
 $\sum_{i=0}^n l_i = -n-1-n$
 $\sum_{i=0}^n l_i = -n-1-n$

The pairing between $H^0(X, \mathcal{O}_X(n))$ and

$H^n(X, \mathcal{O}_X(-n-1-n))$ is given by multiplication:

$$\left(\bigoplus_{\substack{l_i \geq 0 \\ \sum l_i = n}} A X_0^{l_0} \cdots X_n^{l_n} \right) \times \left(\bigoplus_{\substack{m_i < 0 \\ \sum m_i = -n-1-n}} A X_0^{m_0} \cdots X_n^{m_n} \right)$$

$$\xrightarrow{\hspace{15em}} A$$

the dual of $X_0^{l_0} \cdots X_n^{l_n}$ is $X_0^{-l_0-1} \cdots X_n^{-l_n-1}$.

This defines a perfect pairing.

It remains to prove that $H^i(X, \mathcal{F}) = 0$ for $0 < i < n$.

Localize the \check{C} ech complex at X_n : this produces an exact sequence ^{except on the left}, because the cohomology of the localization at X_n is the cohomology of $\mathcal{F}|_{U_n}$ which is 0 because U_n is affine and \mathcal{F} is quasi-coherent.

→ in positive degrees.

$$\Rightarrow \forall i > 0 \quad H^i(X, \mathcal{O}_X)[X_n^{-1}] = 0$$

(localization is exact).

We prove, by induction on n , that multiplication by X_n is injective on $H^i(X, \mathcal{O}_X)$. This will imply

that $H^i(X, \mathcal{O}_X) = 0$ for $0 < i < n$.

$$Z(X_n) \cong \mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^n$$

$$\text{recall } \mathcal{I}_{Z(X_n)} \cong \mathcal{O}_X(-Z(X_n)) \cong \mathcal{O}_X(-1)$$

\Rightarrow exact sequence

$$0 \longrightarrow \mathcal{I}_{Z(X_n)} \xrightarrow{X_n} \mathcal{O}_X \longrightarrow \mathcal{O}_{Z(X_n)} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{X_n} \mathcal{O}_X \longrightarrow \mathcal{O}_Z \longrightarrow 0$$

$$Z = Z(X_n)$$

$$\Rightarrow \quad \forall n$$

$$0 \rightarrow \mathcal{O}_X(-n-1) \xrightarrow{X_n} \mathcal{O}_X(n) \rightarrow \mathcal{O}_Z(n) \rightarrow 0$$

induction hypothesis: $H^i(Z, \mathcal{O}_Z(n)) = 0 \quad \forall n$
 $\forall 0 < i < n-1$

Furthermore, the map

$$H^0(X, \mathcal{O}_X(n)) \rightarrow H^0(Z, \mathcal{O}_X(n))$$

is surjective hom. pol.

in X_0, \dots, X_n of degree n

hom. pol. in
 X_0, \dots, X_{n-1} of degree n .

$$\Rightarrow H^1(X, \mathcal{O}_X(n-1)) \xrightarrow{X_n} H^1(X, \mathcal{O}_X(n)) \text{ is injective}$$

and $H^i(X, \mathcal{O}_X(n-1)) \xrightarrow{X_n} H^i(X, \mathcal{O}_X(n))$ is also

injective $\forall 1 < i < n$ because $H^{i-1}(Z, \mathcal{O}_Z(n)) = 0$
 by induction. □