Fact: If $a_1, \dots, a_r$ is a regular sequence and $I := (a_1, \dots, a_r) \subset A$, then I/I^2 is a free A-module of rank r and, for all d, the natural map

$$\mathrm{Sym}^d(I/I^2) \longrightarrow I^d/I^{d+1}$$

is an isomorphism.

To say that a noetherian local ring of dimension n is Cohen-Macaulay means that there exists a regular sequence $a_1, \dots, a_n$ s.t. the quotient $A/(a_1, \dots, a_n)$ has dimension zero.

(When the ring is regular, $\exists$ sequence s.t. the quotient is a field)

A noetherian local ring of dimension 0 is an Artinian ring (a ring is artinian if it satisfies DCC for chains of ideals, artinian $\iff$ noetherian of dim 0).

- All noetherian local rings of dim. 0 (= local artinian rings) are Cohen-Macaulay.
- One-dimensional reduced noetherian rings are Cohen-Macaulay.
- Two-dimensional normal noetherian rings are Cohen-Macaulay.
- If A is a finitely generated Cohen-Macaulay algebra over a field k with an action of a finite group G, then the subring $A^G \hookrightarrow A$ of invariants

is Cohen-Macaulay.

- Determinantal rings are Cohen-Macaulay.

If A is a ring, we can consider an $m \times n$ matrix with coefficients in A and the ideal generated by its minors of size $r \times s$. ($r \leq m$, $s \leq n$).
Such an ideal is called a determinantal ideal.
A ring B is called determinantal if it is the quotient of a regular local ring by a determinantal ideal.

__Remark:__ When X irreducible and nonsingular over a field k, $\omega_X \cong \Omega^n_{X/k} := \Lambda^n \Omega^1_{X/k}$ where $n = \dim X$.

$H^n(X, \omega_X) \cong H^0(X, \mathcal{O}_X)^*$ is a k-algebra finite-dimensional over k, it is also an integral domain because X is integral $\Rightarrow$ it is a field, i.e., a finite extension of k.

If k is algebraically closed, then

$$H^n(X, \omega_X) \cong H^0(X, \mathcal{O}_X)^* \cong k.$$

Curves. k algebraically closed.

Def: A curve X/k is an integral separated scheme of finite type over k, of dimension 1. We say X is complete if it is proper over k.

Some nice results about curves:

<u>Lemma 1:</u> X nonsingular curve $/k$, $f: X \dashrightarrow Y$ a rational map (i.e., f is a morphism from an open dense subset of X to Y) to a projective variety Y.
Then f extends to a morphism $X \longrightarrow Y$

<u>Proof:</u> Y has a closed embedding $Y \hookrightarrow \mathbb{P}^n_k$.
We can replace Y with $\mathbb{P}^n$ because if we show that f extends to a morphism $X \longrightarrow \mathbb{P}^n$, then the morphism factors through Y, because Y is a closed subscheme of $\mathbb{P}^n$.
Let $U \subset X$ be the largest open set of X where f

is well-defined. The complement

$$X \setminus U = \{P_1, \dots, P_r\} \quad \text{where } P_i \in X \text{ are closed.}$$

$$\begin{tikzcd} f : X \arrow[r, dashed] & \mathbb{P}^n \\ U \arrow[u, hook] \arrow[ur] & \end{tikzcd}$$

The datum of f is equivalent to the data of an invertible sheaf $\mathcal{L}$ on X s.t. $f^*\mathcal{O}_{\mathbb{P}^n}(1) \cong \mathcal{L}|_U$ and a collection of sections $s_0, \dots, s_n$: $s_i = f^* X_i$

$X_i \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$ are the coordinates on $\mathbb{P}^n$.

we know: $U = \{x \in X \mid \exists i \ (s_i)_x \notin m_x \mathcal{L}_x\}$

so $\{P_1, \dots, P_r\} = \{x \in X \mid \forall i \ (s_i)_x \in m_x \mathcal{L}_x\}$

Because X is nonsingular, $\mathcal{O}_{X,x}$ is a DVR $\forall x \in X$ closed point.

$\Rightarrow$ $\mathfrak{m}_{P_i}$ is a principal $\forall i$,

say $\mathfrak{m}_{P_i} = (\pi_i)$

$\forall i, j$, put $n_{i,j} := v_{P_j}((s_i)_{P_j}) > 0$

$n_{i,j}$ is the largest integer s.t. $(s_i)_{P_j} \in (\mathfrak{m}_{P_j})^{n_{i,j}}$

put $n_j := \min_i \{n_{i,j}\}$

and $\mathcal{L}' := \mathcal{L} \otimes \mathcal{O}_X(-n_1 P_1 - \cdots - n_r P_r)$

note that $\mathcal{L}'|_U \simeq \mathcal{L}|_U$

We have the embedding $\mathcal{L}' \hookrightarrow \mathcal{L}$ obtained from $\mathcal{O}_X(-n_1P_1 - \cdots - n_rP_r) \hookrightarrow \mathcal{O}_X$ (1)

by tensoring with $\mathcal{L}$.

The embedding is the embedding of

$\mathcal{O}_X(-P_1) \hookrightarrow \mathcal{O}_X$ as the ideal sheaf of P_1

tensor with $\mathcal{O}_X(-P_1)$ if $n_1 \geq 2$

$\Rightarrow \mathcal{O}_X(-2P_1) \hookrightarrow \mathcal{O}_X(-P_1) \hookrightarrow \mathcal{O}_X$

continue to obtain the embedding (1) above.

locally $\mathcal{O}_X(-n_1P_1 - \cdots - n_rP_r)$ is generated by $\pi_i^{n_i}$

(lift to open neighborhoods)

So if we look at $\underbrace{\mathcal{L} \otimes \mathcal{O}_X(-n_1P_1 - \cdots - n_rP_r)}_{= \mathcal{L}'} \hookrightarrow \mathcal{L}$

$s_i \in H^0(X, \mathcal{L})$

locally, all s_j are divisible by $\pi_i^{n_i}$ $\forall i$, by the definition of the n_i

$\Rightarrow$ via the embedding of $H^0(X, \mathcal{L}') \hookrightarrow H^0(X, \mathcal{L})$ we can consider $s_0, \dots, s_n$ to be global sections of $\mathcal{L}'$ (they belong to the image of $H^0(X, \mathcal{L}')$)

Now map X to $\mathbb{P}^n$ via $\mathcal{L}'$ and $s_0, \dots, s_n \in H^0(X, \mathcal{L}')$

$$\rightsquigarrow \quad \varphi : X \to \mathbb{P}^n$$

$$\forall i \quad \exists j \ \text{ s.t. } \quad (s_j)_{P_i} \notin m_{P_i} \cdot \underbrace{\mathcal{L}'_{P_i}}_{= \pi_i^{n_i} \mathcal{L}_{P_i}}$$

So φ is well-defined everywhere

and $\varphi|_U = f$ because $\mathcal{L}'|_U = \mathcal{L}|_U$ $\square$