($\Rightarrow$ $A \hookrightarrow K(X)$ is the intersection of the valuation rings of $K(X)$ containing it (Atiyah-MacDonald 5.22))

Since X is proper, the valuative criterion implies that $\forall$ $R \hookrightarrow K(X)$ valuation ring, $\exists$ $x \in X$ s.t. $\mathcal{O}_{X,x} \subset R$. Since X is normal ($\Leftrightarrow$ non singular) $\mathcal{O}_{X,x}$ is a D.V.R $\Rightarrow$ $\mathcal{O}_{X,x} = R$

The identity map $A \longrightarrow \mathcal{O}_X(U)$ defines a morphism of schemes $U \to \operatorname{Spec} A$.

<u>Claim:</u> This is an isomorphism.

$\forall$ $x \in U$, we have

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \mathcal{O}_X(U) = A \arrow[d] \\ & \mathcal{O}_{X,x} \supset \mathfrak{m}_x. \end{tikzcd}$$

$x \longmapsto \mathfrak{m} := A \cap \mathfrak{m}_x$ a priori a prime ideal, maximal in this case because A has dim. 1.

now we have

$$\begin{tikzcd}
A_{\mathfrak{m}} \arrow[r, hook] & \mathcal{O}_{X,x} \subset K(X) \\
A \arrow[u, hook] \arrow[ur, hook] &
\end{tikzcd}$$

factorization through localization.

this the morphism of local rings obtained from
$\varphi : U \longrightarrow \operatorname{Spec} A$.

A is an integrally closed domain of dim. 1.

$\Rightarrow$ $A_{\mathfrak{m}}$ is a DVR. $\subset \mathcal{O}_{X,x}$ also a DVR

$\Rightarrow$ $A_{\mathfrak{m}} = \mathcal{O}_{X,x}$. $\Rightarrow$ $\varphi^{\#} : \mathcal{O}_{\operatorname{Spec} A} \xrightarrow{\cong} \varphi_* \mathcal{O}_U$

Provided we prove that φ is a homeomorphism, i.e., it is a bijection.

Surjectivity: Given a maximal ideal $\mathfrak{m}$ of A, $A_{\mathfrak{m}}$ is a DVR, the valuative criterion $\Rightarrow \exists\, x \in X$ s.t. $\mathcal{O}_{X,x} \subset A_{\mathfrak{m}} \subset K(X) \Rightarrow \mathcal{O}_{X,x} = A_{\mathfrak{m}} \Rightarrow \mathfrak{m}_x = \mathfrak{m}$.

Injectivity: $x, y \in U$ s.t. $\mathfrak{m}_x \cap A = \mathfrak{m}_y \cap A \Rightarrow x = y$.

$$\mathfrak{m} = \mathfrak{m}_x \cap A = \mathfrak{m}_y \cap A \Rightarrow \mathcal{O}_{X,x} = \mathcal{O}_{X,y} \subset K(X)$$
$$= A_{\mathfrak{m}}$$

We show that given $x, y \in X$, if $\mathcal{O}_{X,x} = \mathcal{O}_{X,y}$, then $x = y$.

Lemma 2: Suppose X is any quasi-projective integral variety $/k$. Given $x, y \in X$, if $\mathcal{O}_{X,x} = \mathcal{O}_{X,y} \subset K(X)$, then $x = y$.

<u>Proof:</u> Choose an embedding $X \hookrightarrow \mathbb{P}^n_k$.

Replace X with its closure, so we can assume X is projective.

We have $x, y \in X \underset{\text{closed}}{\subset} \mathbb{P}^n_k$

After possibly making a linear change of the coordinates $X_0, \dots, X_n$, we can assume $x, y \in U_0 = D_+(X_0)$

Now replace X with $X \cap U_0 = \mathbb{A}^n_k$.

So we can assume X is an affine variety.

$X = \operatorname{Spec} A \qquad A = k[x_1, \dots, x_n] / I_X$

$x, y \longleftrightarrow$ maximal ideals M_x, M_y of A.

$$\mathcal{O}_{X,x} = A_{M_x} = \mathcal{O}_{X,y} = A_{M_y} \subset K(X) = \operatorname{Frac} A$$

$$\Rightarrow \quad \underset{\substack{\| \\ A \cap M_x A_{M_x}}}{M_x} = \underset{\substack{\| \\ A \cap M_y A_{M_y}}}{M_y} . \quad \Rightarrow \quad x = y .$$

$\square$

<u>Back to the proof of Proposition 2:</u>

We have proved that $f^{-1}(V = \operatorname{Spec} B) = U \cong \operatorname{Spec} A$

where $A = \mathcal{O}_X(U)$.

It remains to prove that the extension of rings $A \subset B$ is finite (or integral).

Since X is proper $/k$ and Y is separated $/k$ (in fact complete), we deduce that X is proper over Y (II.4).

The valuative criterion implies that for any valuation ring R of $K(X)$ containing B, $\exists\ x \in X$ whose local ring contains R, i.e., $\mathcal{O}_{X,x} = R$ as before.

Therefore $R = \mathcal{O}_{X,x}$ also contains A.

Hence any valuation ring of $K(X)$ containing B contains A

$$\Rightarrow \quad \underbrace{A \subset \bigcap_{R \supset B} R}_{\text{the integral closure of } B \text{ in } K(X)} \subset K(X)$$

$$A \text{ is integrally closed} \Rightarrow \quad A = \bigcap_{R \supset B} R \subseteq K(X).$$

$\Rightarrow$ A is the integral closure of B in $K(X)$.

$\Longrightarrow$ The extension $B \subset A$ is integral, i.e., finite. $\square$.

<u>Definition</u>: Suppose $f : X \longrightarrow Y$ is a finite morphism of nonsingular curves $/k$. We define a homomorphism

$$f^* : \mathrm{Div}(Y) \longrightarrow \mathrm{Div}(X) \text{ as follows.}$$

For any point $Q \in Y$, let $t \in \mathcal{O}_{Y,Q} \subset K(Y)$ be a uniformizer or local parameter, i.e., a generator of the maximal ideal $m_Q \subset \mathcal{O}_{Y,Q} \subset K(Y)$. Define

$$f^*[Q] := \sum_{f(P)=Q} v_P(t)[P] \quad \in \mathrm{Div}(X)$$

and extend by linearity to all Weil divisors.

The definition makes sense:

(1) the sum is finite because we saw last quarter that finite morphisms are quasi-finite, i.e., $\forall\, Q \in Y$. the number of points of X mapping to Q is finite.

(2) the definition does not depend on the choice of t, because any other uniformizer differs from t by multiplication by an invertible element of $\mathcal{O}_{Y,Q}$.

$\forall \quad P$ mapping to Q, we have
$$f^{\#}: \mathcal{O}_{Y,Q} \longrightarrow \mathcal{O}_{X,P}$$
$$t \longmapsto f^{\#}(t)$$

units of $\mathcal{O}_{Y,Q}$ map to units of $\mathcal{O}_{X,P}$ which have valuation 0.

Some nice properties of f^*:

<u>Exercise</u>: (1) Pull-backs of principal divisors are again principal, so that we obtain an induced morphism

$$f^*: Cl(Y) \longrightarrow Cl(X).$$

This coincides with pull-backs of invertible sheaves and the pull-back $f^*: CaCl(Y) \longrightarrow CaCl(X)$ under the identifications $Cl(X) = CaCl(X) = Pic(X)$

(2) <u>Proposition 3:</u> $f: X \to Y$ finite morphism of integral nonsingular curves. Then for any divisor D on Y, we have $\deg(f^* D) = \deg(f) \deg D$.

Recall that if $D = \sum n_P [P]$, then $\deg D = \sum n_P$.

and $\deg f = \deg K(X)/K(Y)$.

<u>Proof:</u> By linearity, we need to show that for $Q \in Y$,

$$\deg f^* Q = \deg f.$$

As in the proof of Proposition 2, let $V = \operatorname{Spec} B$ be an affine open set of Y, and put $U = f^{-1}(V) = \operatorname{Spec} A$.

We know that A is the integral closure of B in $K(X)$.

Let $\mathfrak{m}$ be the maximal ideal of B corresponding to Q and localize B and A at $\mathfrak{m}$ to obtain

$$\mathcal{O}_{Y,Q} = B_{\mathfrak{m}} \qquad A' := A \otimes_B B_{\mathfrak{m}}$$

Since B_m is a DVR (in particular it is a PID) and A' is torsion-free, we obtain that A' is a free B_m-module of finite rank. The rank of A' over B_m is $r := \deg f$ because if we further localize B_m and A' at all nonzero elements of B_m we obtain the field extension $K(Y) \subset K(X)$. (the localization of A' is a finite ring extension of $K(Y)$ which is an integral domain $\underset{\text{comm. alg.}}{\Rightarrow}$ it is a field $= K(X) = \mathrm{Frac}(A')$).

Furthermore, if t is a uniformizer at Q, then

$$A'/tA' \text{ is a vector space over } \mathcal{O}_{Y,Q}/t\mathcal{O}_{Y,Q} = k(Q) = k$$

of dim. r.

The points P_i of X mapping to Q are in one-to-one correspondence with the maximal ideals of A pulling back to $\mathfrak{m}$, i.e., containing $\mathfrak{m} \subset B \subset A$, these are in bijection with the maximal ideals $\mathfrak{m}_i$ of A'.