The points P_i of X mapping to Q are in one-to-one correspondence with the maximal ideals of A pulling back to m, i.e., containing $m \subset B \subset A$, these are in bijection with the maximal ideals m_i of A'.

<u>Now</u>: $A' = \bigcap_i A'_{m_i}$ because A' is an integrally closed domain.

$$\Rightarrow \quad tA' = \bigcap_i \left(tA'_{m_i} \cap A' \right)$$

By the Chinese remainder theorem:

$$A'/tA' \cong \bigoplus_i A'/tA'_{m_i} \cap A' = \bigoplus_i A'_{m_i}/tA'_{m_i}$$

(map $A' \longrightarrow A'_{m_i} \twoheadrightarrow A'_{m_i}/tA'_{m_i}$ and factor through $tA'_{m_i} \cap A'$ to get the second isom.)

$$= \bigoplus_i \mathcal{O}_{P_i} / t\,\mathcal{O}_{P_i}$$

$$\Rightarrow \deg f = n = \operatorname{rank} A' = \dim_k A'/tA' = \sum_i \dim \mathcal{O}_{P_i}/t\,\mathcal{O}_{P_i}$$

$$= \sum_i \nu_{P_i}(t)$$

$$= \deg f^* Q \text{ by def.}$$

$\square$.

<u>Corollary</u>: The degree of a principal divisor is 0 (on a complete nonsingular curve).

<u>Proof</u>: Let $f \in K(X)$. If $f \in k$, then $\mathrm{Div}(f) = 0$ and we are done. If $f \notin k$, then, since k is algebraically closed, f is transcendental over k, hence

f defines a nonconstant morphism $\varphi : X \to \mathbb{P}^1$ as follows:

Write $\mathrm{Div}(f) = \sum_i m_i P_i - \sum n_j Q_j$

where all the integers m_i and n_j are nonnegative and $\{P_i\} \cap \{Q_j\} = \emptyset$.

Put $U := X \setminus \{Q_j\}$, $U' := X \setminus \{P_i\}$.

$\forall\ P \in U$, $f \in \mathcal{O}_{X,P} \Rightarrow f \in \mathcal{O}_X(U) = \bigcap_{P \in U} \mathcal{O}_{X,P}$.

$\Rightarrow \exists$ morphism $\varphi_U : U \to \mathbb{A}^1$

defined by $\mathcal{O}_X(U) \hookleftarrow k[x]$

$f \longmapsfrom x$

Similarly define $\varphi_{U'} : U' \to \mathbb{A}^1$ using $\frac{1}{f}$

These glue to $\varphi : X \to \mathbb{P}^1$. $\quad X = U \cup U'$

We have $\mathrm{Div}(f) = \varphi^*(\{0\} - \{\infty\})$

has degree $0 = (\deg \varphi) \deg(\{0\} - \{\infty\})$. $\square$

Exercise: In the proof above, show that

$$\varphi^* \mathcal{O}_{\mathbb{P}^1}(1) \cong \mathcal{O}_X\left(\sum_i m_i P_i\right) \cong \mathcal{O}_X\left(\sum_j n_j Q_j\right).$$

More nice results about curves: X nonsingular complete$/k$

Lemma 3: D a divisor on X. If $h^0(D) := \dim_k H^0(D) \neq 0$, then $\deg D \geq 0$. If $h^0(D) \neq 0$ and $\deg D = 0$, then $D \sim 0$, i.e., $\mathcal{O}_X(D) \cong \mathcal{O}_X$.

Proof: If $h^0(D) \neq 0$, then for $s \in H^0(D)$, $s \neq 0$

$\mathrm{Div}(s)$ is an effective divisor, $= \mathrm{Div}(\mathcal{Z}(s))$

$\Rightarrow \quad \deg \operatorname{Div}(s) \geq 0$

$\mathcal{O}_X(D - \operatorname{Div}(s)) \cong \mathcal{O}_X \Rightarrow D - \operatorname{Div}(s)$ is principal

$\Rightarrow \deg D = \deg \operatorname{Div}(s) \geq 0.$

If $\deg D = 0$, then $D \sim \operatorname{Div}(s)$ effective of degree 0

$\Rightarrow \operatorname{Div}(s) = 0 \quad \Rightarrow D \sim 0 \Rightarrow \mathcal{O}_X(D) \cong \mathcal{O}_X$ $\square$.

<u>Definition:</u> For a projective curve, its arithmetic genus is defined to be (X not necessarily nonsingular)

$$p_a := h^1(X, \mathcal{O}_X) := \dim H^1(X, \mathcal{O}_X).$$

The geometric genus is defined to be

$$g := h^0(X, \Omega^1_X).$$

When X is nonsingular, then $\omega_X \cong \Omega^1_X$, and, by Serre duality, $\forall$ quasi-coherent sheaves $\mathcal{F}$:

$$H^0(X, \mathcal{F}) \cong \mathrm{Ext}^1_X(\mathcal{F}, \omega_X)^*$$

$$H^1(X, \mathcal{F}) \cong \mathrm{Ext}^0_X(\mathcal{F}, \omega_X)^* \cong \mathrm{Hom}_X(\mathcal{F}, \omega_X)^*$$

If $\mathcal{F}$ is locally free of finite rank:

$$H^0(X, \mathcal{F}) \cong H^1(X, \omega_X \otimes \mathcal{F}^*)^*$$

$$H^1(X, \mathcal{F}) \cong H^0(X, \omega_X \otimes \mathcal{F}^*)^*.$$

<u>Theorem</u>: Riemann-Roch for curves:

X complete nonsingular curve of genus g.

D a divisor on X. Then

$$\chi(D) := h^0(D) - h^1(D) = \deg D + 1 - g.$$

<u>Proof:</u> Let P be any (closed) point of X.

We show that the theorem is true for D iff it is true for $D-P$. Starting from 0 and adding and subtracting points, this will prove the theorem for all divisors (for $D=0$, $\chi(0) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) = 1-g$
$= \deg 0 + 1 - g$).

Recall that the ideal sheaf of $P \hookrightarrow X$ is isom. to $\mathcal{O}_X(-P)$.

This gives us the exact sequence:

$$0 \longrightarrow \mathcal{O}_X(-P) \longrightarrow \mathcal{O}_X \longrightarrow \underset{\substack{\| \\ i_* \mathcal{O}_P}}{\mathcal{O}_P} \longrightarrow 0$$

(abuse of notation always done $\quad i : P \hookrightarrow X$)

twist by $\mathcal{O}_X(D)$ to obtain:

$$0 \longrightarrow \mathcal{O}_X(D-P) \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_P(D) \longrightarrow 0$$

Note that $\mathcal{O}_P(D) \cong \mathcal{O}_P$ because $P \cong \operatorname{Spec} k$

= skyscraper sheaf supported at P with group k.

From the long exact sequence of cohomology, we obtain: $\chi(D-P) = \chi(D) - 1$.

Since we also have $\deg(D-P) = \deg D - 1$,

we obtain $\chi(D-P) - \deg(D-P) = \chi(D) - \deg D$,

and we are done. $\square$

<u>Corollary</u>: The degree of the canonical sheaf $\omega_X = \Omega^1_X$ is $2g-2$.

Proof: Let K_X be a divisor s.t. $\mathcal{O}_X(K_X) \cong \omega_X$.

Then $\chi(K_X) = \deg K_X + 1 - g$

$$\begin{aligned} &\| \\ &h^0(K_X) - h^1(K_X) = h^0(\omega_X) - h^1(\omega_X) \\ &\text{(Serre Duality)} = \underbrace{h^1(\mathcal{O}_X)}_{\substack{\| \\ g}} - h^0(\mathcal{O}_X) = g-1 \end{aligned}$$

$$\Rightarrow \deg K_X = g-1 + g-1 = 2g-2 \qquad \square.$$

Def: A canonical divisor is a divisor K_X s.t.

$$\mathcal{O}_X(K_X) \cong \omega_X \cong \Omega^1_X.$$

Definition: We say that D is special if $h^1(D) \neq 0$, and D is nonspecial if $h^1(D) = 0$.

Remark: (1) If $\deg D \geq 2g-1$, then D is nonspecial.

$$\deg D \geq 2g-1 \Rightarrow \deg(K_X - D) = \deg K_X - \deg D$$
$$= 2g-2-\operatorname{Deg} D \leq -1 < 0$$

$\Rightarrow h^0(K_X - D) = 0$ by Lemma 3.

$= h^1(D)$ by Serre Duality.

(2) If D is special of degree $2g-2$, then $\mathcal{O}_X(D) \cong \omega_X$.

(apply Lemma 3 to $K_X - D$)

Theorem: Kodaira vanishing (see Griffiths-Harris)

X projective nonsingular of dim. n over $\mathbb{C} = k$.

$\mathcal{L}$ ample invertible sheaf on X.

The following equivalent statements hold:

(1) $H^i(X, \mathcal{L} \otimes \omega_X) = 0 \quad \forall\, i > 0$

(2) $H^i(X, \mathcal{L}^{-1}) = 0 \quad \forall\, i < n$.

<u>Proof for curves:</u> Write $\mathcal{L} = \mathcal{O}_X(D)$. If D is ample, then a multiple is very ample.

In particular, $\exists\ m > 0$ s.t. $h^0(mD) \geq 2$.

Lemma 3 $\Longrightarrow \deg D > 0 \Rightarrow \deg(K_D + D) \geq 2g-1$

$\Rightarrow h^1(K_X + D) = 0$, i.e., $K_X + D$ is nonspecial. $\square$.

We shall soon see that on a curve, a divisor is ample iff it has positive degree.

Definition: A variety is a separated integral scheme of finite type $/k$.

Two varieties X, Y are birational if $\exists$ nonempty open sets $U \subset X$, $V \subset Y$ s.t. $U \cong V$.

A variety X is called rational if it is birational to $\mathbb{P}^n$ where $n = \dim X$. (This is equivalent to

$$K(X) \cong K(\mathbb{P}^n) \cong k(x_1, \dots, x_n))$$

Proposition: A complete nonsingular curve X is rational iff it is isomorphic to $\mathbb{P}^1$, iff it has genus 0.

Proof: If X has genus 0, then Riemann-Roch becomes: $\chi(D) = \deg D + 1$.

Choose two distinct points $P, Q \in X$, then

$$\chi(P-Q) = 1 = h^0(P-Q) - h^1(P-Q)$$

$$\Rightarrow \quad h^0(P-Q) > 0$$

Lemma 3 $\Longrightarrow$ $P - Q \overset{\text{lin.}}{\sim} 0$

Choose a rational function $f \in K(X)$ s.t.

$$\mathrm{Div}(f) = P - Q.$$

As in the proof of the Corollary to Prop. 3, f gives a morphism φ to $\mathbb{P}^1$ s.t. $\mathrm{Div}(f) = \varphi^*(\{0\} - \{\infty\})$ and $\varphi^*\{0\} = [P]$ $\Longrightarrow$ $\deg \varphi = 1$ $\Longrightarrow$ φ is an isom. $\square$