Next case: Elliptic curves.

Definition: A smooth (= nonsingular) complete curve is called elliptic if it has genus 1.

Some facts about elliptic curves:

(1) The canonical sheaf has degree $2g-2=0$. Since the space of global sections $H^0(\omega_X)$ has dim. 1, we deduce that $\omega_X = \mathcal{O}_X(K_X)$ is trivial:

$$\omega_X \cong \mathcal{O}_X.$$

(2) Fix a closed point $x_0 \in X$. Then the map

$$X \longrightarrow \mathrm{Pic}^0(X) \subset \mathrm{Pic}(X)$$

subgroup of degree 0 divisors

$$x \longmapsto \mathcal{O}_X(x - x_0)$$

is a bijection.

<u>Proof</u>: First represent any $\mathcal{L} \in \mathrm{Pic}^0 X$ by a divisor D of degree 0: $\mathcal{L} \cong \mathcal{O}_X(D)$.

For any divisor D of degree 0, the divisor $D + x_0$ has degree $1 = 2g - 1$, hence it is nonspecial, i.e., $h^1(D + x_0) = 0$. Now Riemann-Roch implies $h^0(D + x_0) = 1$, i.e., $H^0(\mathcal{O}_X(D + x_0))$ has dim. 1.

The divisor of zeros of a nonzero section of $\mathcal{O}_X(D + x_0)$ is the unique effective divisor of degree 1 linearly equivalent to $D + x_0$, i.e., $\exists!\ x \in X$ s.t.

$$D + x_0 \sim x. \qquad \square$$

Via the bijection $X \longrightarrow \mathrm{Pic}^0(X)$
$x \mapsto \mathcal{O}_X(x - x_0)$

we can define a group structure on the set of closed points of X. The group structure depends on the choice of x_0:
x_0 is its origin.

We say that X is a principal homogeneous space or a torsor on $\mathrm{Pic}^0(X)$.

This is similar to the situation of an affine space and its associated vector space.

Residues and the trace map of Serre Duality:

A small reminder about completions:

R local ring with maximal ideal m.

The completion of R is $\hat{R} := \varprojlim_n R/m^n$.

When R is a k-algebra, noetherian, regular of dim. n,

then $\hat{R} \cong k[[x_1, \dots, x_n]]$

the ring of formal power series in n variables over k.

The field of fractions of $\hat{R}$ is isomorphic to

$$k((x_1, \dots, x_n))$$

the field of Laurent power series over k. (see comm. alg. books)

<u>Residues</u>: X smooth complete curve $/k$ as before.

Recall $\omega_X \cong \Omega^1_{X/k}$ the sheaf of Kähler differentials.

The sheaf $\omega_X \otimes K(X) := \omega_X \otimes_{\mathcal{O}_X} \mathcal{K}_X \ (\cong \mathcal{K}_X)$ is the sheaf of rational (or meromorphic) differential forms on X.

For a closed point $x \in X$, let $t \in \mathcal{O}_{X,x} \subset K(X)$ be a uniformizer. Then $\widehat{\mathcal{O}}_{X,x} = k[[t]]$, and

$$\omega_{X,x} = \Omega^1_{\mathcal{O}_{X,x}/k} = \mathcal{O}_{X,x}\, dt, \text{ hence}$$

$$\widehat{\omega}_{X,x} = \widehat{\mathcal{O}}_{X,x}\, dt = k[[t]]\, dt$$

and $\widehat{\omega_{X,x} \otimes K(X)} = k((t))\, dt$, and the germ at x of any section of $\omega_X \otimes K(X)$ has a Laurent expansion

$$\left(\frac{a_{-n}}{t^n} + \frac{a_{-n+1}}{t^{n-1}} + \cdots + \frac{a_{-1}}{t} + \sum_{i \geq 0} a_i t^i \right) dt.$$

<u>Definition:</u> The residue of $w \in H^0(\omega_X \otimes K(X))$ at x is the coefficient a_{-1} in the Laurent expansion of ω_x at x. It is denoted $\mathrm{Res}_x(w)$.

One can show that $\mathrm{Res}_x(w)$ is independent of the choice of the uniformizer t. (references in III.7)

As in the case of rational functions, the number of poles (or zeros) of a meromorphic differential form

$$w \in H^0(\omega_X \otimes K) \cong H^0(\mathcal{K}_X)$$

is finite. So $\mathrm{Res}_x(w)$ is nonzero at most for finitely many $x \in X$.

<u>Theorem</u> (Residue theorem):

For all $w \in H^0(\omega_X \otimes K)$:

$$\sum_{x \in X} \mathrm{Res}_x(w) = 0$$

<u>Proof</u>: See references in Hartshorne.

<u>The trace map of Serre Duality for curves</u>:

We have the exact sequence:

$$0 \to \mathcal{O}_X \to \mathcal{K}_X \to \mathcal{K}_X/\mathcal{O}_X \to 0$$

One can show (exercise):

$$\mathcal{K}_X/\mathcal{O}_X \cong \bigoplus_{x \in X} K/\mathcal{O}_{X,x}$$

where $K/\mathcal{O}_{X,x}$ is the skyscraper sheaf at x with group $K/\mathcal{O}_{X,x}$.

Tensor with ω_X to get the exact sequence

$$0 \to \omega_X \to \omega_X \otimes \mathcal{K}_X \to (\omega_X \otimes \mathcal{K}_X)/\omega_X \to 0$$

with long exact sequence of cohomology:

$$0 \to H^0(\omega_X) \to H^0(\omega_X \otimes \mathcal{K}_X) \to H^0((\omega_X \otimes \mathcal{K}_X)/\omega_X) \to H^1(\omega_X) \to 0$$

Note that $H^1(\omega_X \otimes \mathcal{K}_X) = H^1(\mathcal{K}_X) = 0$.

and $H^1((\omega_X \otimes \mathcal{K}_X)/\omega_X) = 0$ because these are flasque sheaves (exercise).

We have the residue map

$$H^0(\omega_X \otimes \mathcal{K}_X/\omega_X) = \bigoplus_{x \in X} (\omega_{X,x} \otimes K)/\omega_{X,x} \xrightarrow{\sum_{x\in X} \mathrm{Res}_x} k$$

vanishes on the image of $H^0(\omega_X \otimes K)$ by the Residue theorem. So, from the long exact sequence of cohomology, the residue factors through a map

$$H^1(\omega_X) \xrightarrow[\sim]{Tr} k$$

which is the trace map of Serre Duality in this case.

<u>The Hurwitz theorem:</u>

<u>Some preparation:</u>

<u>Definition:</u> Let $f: X \longrightarrow Y$ be a finite morphism of nonsingular curves. Recall $\deg f := [K(X) : K(Y)]$.

For a point $x \in X$, let y be its image in Y.

The ramification index of f at x is

$$e_x := v_x(t_y)$$

where t_y is a uniformizer at y. Note that $e_x \geq 1 \ \forall x$.

We say that f is ramified at x if $e_x \geq 2$, otherwise, we say f is unramified at x.

If $\mathrm{char}\, k = 0$ or if $\mathrm{char}\, k \nmid e_x$, we say the ramification is tame at x, otherwise, we say it is wild.

With the above definition, the map $f^*: \mathrm{Div}(Y) \to \mathrm{Div}(X)$ can be written as $f^*([y]) = \sum_{x \mapsto y} e_x [x]$.

Recall that $f^* \mathcal{O}_Y(D) \cong \mathcal{O}_X(f^*D)$ (exercise) so that f^* on Div induces f^* on Pic.

Def: We say f is separable if $K(Y) \subset K(X)$ is separable.

Proposition: $f: X \to Y$ a finite separable morphism of nonsingular curves. Then we have the exact sequence

$$0 \longrightarrow f^*\Omega^1_{Y/k} \longrightarrow \Omega^1_{X/k} \longrightarrow \Omega^1_{X/Y} \longrightarrow 0$$

Proof: The sequence exists in general (203B) and is exact except possibly on the left.
To see exactness on the left, since both sheaves are invertible, it suffices to see that the map $f^*\Omega^1_Y \to \Omega^1_X$ is nonzero (203B homework). To see this, we localize to

the generic point of X where the stalk of $\Omega^1_{X/Y}$ is $\Omega^1_{K(X)/K(Y)}$.

We have $\Omega^1_{K(X)/K(Y)} = 0$ because $K(X)/K(Y)$ is separable. To check this, write $K(X) = K(Y)[\alpha]$.

α algebraic $/K(Y)$, i.e.,

$$K(X) \cong K(Y)[T] \big/ P(T)$$

T variable
P polynomial with coeff in $K(Y)$

$$\Rightarrow \quad \Omega^1_{K(X)/K(Y)} \cong \left(K(Y)[T] \big/ P(T) \right) dT \Big/ P'(T)\, dT$$

$$\Rightarrow \quad \Omega^1_{K(X)/K(Y)} \cong \frac{K(Y)[T]\, dT}{(P(T)dT,\ P'(T)\, dT)}$$

Since the extension is separable, P & P' are coprime

$$\Rightarrow \quad \Omega^1_{K(X)/K(Y)} = 0$$

Therefore the morphism $f^*\Omega^1_Y \to \Omega^1_X$ is surjective at the generic point, hence not the zero morphism. $\square$

Note: Since $\Omega^1_{X/Y}$ is coherent, the fact that its stalk at the generic point is zero means that $\Omega^1_{X/Y}$ is zero on some open subset of X, i.e., $(\Omega^1_{X/Y})_x$

is nonzero only at finitely many points of X.