Because the target is 1-dimensional, surjectivity means the map is not 0. This means $\exists\ s \in H^0(D)$ s.t. $s_x \in m_x \mathcal{O}_X(D)_x$ and $s_x \notin m_x^2 \mathcal{O}_X(D)_x$.

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$x \in \operatorname{Supp} Z(s)$

(if $D(s)$ = divisor of s)

$\Leftrightarrow$ $D(s) - x \geq 0$ $\qquad\qquad$ $D(s) - 2x$ not effective.

$$\Updownarrow \qquad\qquad\qquad\qquad \Updownarrow$$

$s \in H^0(D-x)$ $\qquad\qquad$ $s \notin H^0(D-2x)$

(use sequence $0 \to \mathcal{O}_X(-2x) \to \mathcal{O}_X \to \mathcal{O}_{\{2x\}} \to 0$)

This means that $H^0(D-2x) \underset{\neq}{\hookrightarrow} H^0(D-x) \hookrightarrow H^0(D)$

$\Leftrightarrow$ $h^0(D-2x) = h^0(D-x) - 1 = h^0(D) - 2$ $\qquad \square$

<u>Definition:</u> The degree of a morphism $X \xrightarrow{\varphi} \mathbb{P}^n_k$ is the degree of the invertible sheaf $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$.

This is equal to the degree of any divisor

D s.t. $\mathcal{O}_X(D) \cong \varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$

which is also the degree of the divisor of zeros of any global section of $\varphi^* \mathcal{O}_{\mathbb{P}^n}(1)$, in particular, for any $H \in H^0(\mathcal{O}_{\mathbb{P}^n}(1))$, it is the degree of $D(\varphi^* H)$. (see Chapter 1 Section 7 for more on degrees of morphisms)

<u>Corollary:</u> (1) $\deg D \geq 2g \Rightarrow |D|$ is base point free.

(2) $\deg D \geq 2g+1 \Rightarrow |D|$ is very ample.

(3) D is ample $\iff$ $\deg D > 0$.

<u>Proof</u>: exercise.

<u>Some other nice consequences!</u>

- We saw that $X \cong \mathbb{P}^1 \iff g=0 \iff K(X) \cong K(\mathbb{P}^1) \cong k(t)$
  In this case: ample $\iff$ very ample $\iff$ $\deg D > 0$
- If X is elliptic, i.e., $g=1$, then any divisor of degree 2 is base point free and $h^0(D)=2$ (use Riemann-Roch and $h^1(D)=0$)

so $|D|$ defines a morphism of degree 2

$$X \twoheadrightarrow \mathbb{P}^1.$$

If $\deg D = 3$, then $|D|$ is very ample and $h^0(D)=3$, and $|D|$ defines a closed embedding $X \hookrightarrow \mathbb{P}^2$ of degree 3.

The image of X is a plane cubic.

For elliptic curves, very ample $\iff \deg D \geqslant 3$.

<u>Corollary:</u> The canonical linear system $|K_X| = |\omega_X| = |\Omega^1_{X/k}|$ is base point free iff $g > 0$.

<u>Proof:</u> $|K_X|$ is base point free iff $\forall\, x \in X$

$$h^0(K_X - x) = h^0(K_X) - 1 = g - 1.$$

$$h^0(x) \underset{\text{Serre Duality}}{=} h^1(K_X - x) \underset{\text{Riemann–Roch}}{=} h^0(K_X - x) + g - 1 - \deg(K_X - x)$$

$$= h^0(K_X - x) + g - 1 - (2g - 3) = h^0(K_X - x) - g + 2$$

$$\leq g - g + 2 = 2$$

$h^0(x)$ cannot be 2 because otherwise it would give a morphism of degree 1, $X \to \mathbb{P}^1$ which would then be an isom, but $g > 0$.

$$\Rightarrow \quad h^0(x) = 1 = h^0(K_X - x) - g + 2$$

$$\Rightarrow \quad h^0(K_X - x) = g - 1 \qquad \square$$

- <u>Curves of genus 2:</u>

If X has genus 2, then $\deg K_X = 2g-2 = 2$ and $h^0(K_X) = g = 2$. So $|K_X|$ defines a morphism of degree 2 : $X \twoheadrightarrow \mathbb{P}^1$.

Any divisor of degree ≥ 5 is very ample.

One can show that D very ample $\Leftrightarrow$ $\deg D \geq 5$.

<u>Definition:</u> We say that X is hyperelliptic if $\exists\ \varphi : X \twoheadrightarrow \mathbb{P}^1$ of degree 2.

We say X is trigonal if $\exists\ \varphi : X \twoheadrightarrow \mathbb{P}^1$ of degree 3.

" " " " tetragonal " " " " 4.

$\vdots$

" n-gonal " " n.

A linear system of degree d and dim r is called (and denoted) a g^r_d. (e.g. if $\deg D = 2$ and $\dim |D| = 1$, then $|D|$ is a g^1_2).

$\overset{\shortparallel}{h^0(D)-1}$

We saw that curves of genus ≤ 2 are alway hyperelliptic.

Theorem: The canonical linear system $|K_X|$ is very ample if and only if X is NOT hyperelliptic.

Proof: $|K_X|$ is very ample iff $\forall\ p, q \in X$

$$h^0(K_X - p - q) = h^0(K_X) - 2 = g - 2.$$

By Serre Duality this is equivalent to $h^1(p+q) = g-2$

and by Riemann-Roch, this is equivalent to $h^0(p+q) = 1$.

So $|K_X|$ is not very ample iff $\exists\ p, q \in X$

s.t. $h^0(p+q) \geqslant 2$.

Note that if $g=0$, $X \cong \mathbb{P}^1$, $\omega_{\mathbb{P}^1} \cong \mathcal{O}_{\mathbb{P}^1}(-2) \cong \mathcal{O}(K_{\mathbb{P}^1})$

cannot be very ample.

If $g>0$ $\quad h^0(p+q) = h^1(K_X-p-q) \leq 2$.

So K_X not very ample $\Leftrightarrow$ $\exists\, p,q \in X$ s.t. $h^0(p+q)=2$.

Then $|p+q|$ is a g^1_2 and has no base points because $h^0(p+q-x) \leq 1$ for any divisor of degree 1 because as before, if not we would have $X \cong \mathbb{P}^1$.

So $|p+q|$ defines a morphism of degree 2 to $\mathbb{P}^1$

$\Leftrightarrow$ X is hyperelliptic. $\square$

<u>Definition:</u> If X is not hyperelliptic (in particular $g \geq 3$), then the embedding of X in $\mathbb{P}^{g-1}_k$ is called the canonical embedding and the image of X by the

canonical embedding is called the canonical curve (which is isomorphic to X).

- If X has genus 3 and is not hyperelliptic, then the canonical curve $X \hookrightarrow \mathbb{P}^2_k$ is a smooth plane quartic. $\deg K_X = 2g-2 = 4$. (see I.7)
Conversely, any smooth plane quartic is a canonical curve. $\Rightarrow$ non hyperelliptic curves of genus 3 exist.
- If X has genus 4 and is not hyperelliptic, then the canonical curve $X \hookrightarrow \mathbb{P}^3_k$ is the complete intersection of a unique irreducible quadric and an irreducible cubic in $\mathbb{P}^3$, i.e., the homogeneous ideal of X

in $\mathbb{P}^3$ is generated by an irreducible quadratic polynomial (unique up to multiplication by a scalar) and an irreducible cubic polynomial.

Sketch of proof: $I_X = \bigoplus_{d \geq 0} H^0(\mathcal{I}_X(d))$

$$0 \longrightarrow \mathcal{I}_X \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_X \longrightarrow 0$$

We know that homogeneous polynomials of degree d vanishing on X are the elements of $H^0(\mathcal{I}_X(d))$.

$$\begin{array}{ccccccccc}
0 \to & H^0(\mathcal{I}_X) & \to & H^0(\mathcal{O}_{\mathbb{P}^3}) & \xrightarrow{\quad} & H^0(\mathcal{O}_X) & \to & \cdots \\
& \| & & \cong & \hookrightarrow & \cong & & \\
& 0 & & k & & k & &
\end{array}$$

twist by $\mathcal{O}_{\mathbb{P}^3}(1)$ and take cohomology:

$$0 \to H^0(\mathcal{I}_X(1)) \longrightarrow H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \overset{\cong}{\twoheadrightarrow} H^0(\mathcal{O}_X(1))$$