Surfaces:

In this section, by a surface we mean a nonsingular projective surface over an algebraically closed field.

Now divisors are supported on curves.
As usual, the datum of an effective divisor is equivalent to the datum of a closed subscheme of codim 1.

We start by intersecting divisors on a surface:

We define a bilinear pairing on $Cl(X) \cong CaCl(X) \cong Pic(X)$ which is an extension of the naïve notion of intersecting transverse curves on a surface.
Recall that for two closed subschemes Y and Z of a scheme X, the intersection scheme $Y \cap Z$ is the closed

subscheme with sheaf of ideals $\mathcal{I}_Y + \mathcal{I}_Z \hookrightarrow \mathcal{O}_X$.

<u>Definition:</u> We say that two curves (closed subschemes of codim. 1) C and D on X (a surface) intersect transversely at a (closed) point $x \in C \cap D$ if the local equations of C and D at x generate the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$.

This implies, in particular, that C and D are both nonsingular at x (for instance, the local equation of D at x will generate the maximal ideal of $\mathcal{O}_{C,x}$).

We say that C and D are transverse if they are transverse at all of their points of intersection.

__Theorem__: There exists a unique symmetric bilinear form

$$\mathrm{Div}(X) \times \mathrm{Div}(X) \longrightarrow \mathbb{Z}$$
$$(C, D) \longmapsto C \cdot D$$

such that, if C and D are transverse curves, then $C \cdot D = \# C \cap D$ and, if $C \sim C'$ (linear equivalence) then, for all $D \in \mathrm{Div}(X)$, $C \cdot D = C' \cdot D$.

__Lemma__: Suppose $C \subset X$ is a nonsingular curve and $D \subset X$ is any curve meeting C transeversely. Then

$$\# C \cap D = \deg \mathcal{O}_C(D).$$

__Proof__: We have the usual exact sequence

$$0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_{C\cap D} \longrightarrow 0.$$
$$\mathcal{O}_C(-D) = \mathcal{O}_C(-C\cap D)$$

Since C and D are transverse, their intersection is reduced, hence $\mathcal{O}_{C\cap D}$ has length $\# C\cap D$.

The exact sequence above now gives:

$$\deg \mathcal{O}_C(-D) = -\deg \mathcal{O}_C(D) = -\deg \mathcal{O}_{C\cap D}$$
$$\Rightarrow \quad \deg \mathcal{O}_C(D) = \# C\cap D. \qquad \square$$

<u>Remark:</u> From the lemma, we can extend the intersection pairing by linearity to any pairs of transverse curves that do not have any common (irreducible) components.

In this more general case we still obtain

$$C \cdot D = \deg \mathcal{O}_C(D) = \operatorname{length} \mathcal{O}_{C \cap D} = \sum_{x \in C \cap D} \operatorname{length}_x \mathcal{O}_{C \cap D, x}$$

$$= \# C \cap D .$$

If the curves C and D have no common component but are not necessarily transverse, we obtain

$$C \cdot D = \deg \mathcal{O}_C(D) = \sum_{x \in C \cap D} \operatorname{length}_x \mathcal{O}_{C \cap D, x}$$

$$= \sum_{x \in C \cap D} \operatorname{length}_x \left(\mathcal{O}_{X, x} / (f_x, g_x) \right)$$

where f_x, g_x are the local equations of C and D at x.

Proof of theorem about the existence and uniqueness of the intersection pairing:

Uniqueness: Choose a very ample divisor H on X, and let $C, D \in \mathrm{Div}(X)$ be arbitrary.

$\exists\ m > 0$ s.t. $C + mH$ and $D + mH$ are generated by global sections. By homework from last quarter (Ex. II.7.5), $C + (m+1)H$, $D + (m+1)H$ are very ample.

Replace H by $(m+1)H$ so that $C+H$ and $D+H$ are very ample.

Now apply Bertini's theorem: $\exists\ C' \in |C+H|$ which is nonsingular and irreducible.

Apply Bertini twice more: $\exists$ open set U of $|D+H|$ of curves that are nonsingular and irreducible,

$\exists$ open set V of $|\mathcal{O}_{C'}(D+H)|$ of reduced divisors.

Choose $D' \in U \cap \pi^{-1}(V)$ where

$$\pi : |D+H| \longrightarrow |\mathcal{O}_{C'}(D+H)|$$

is the restriction map.

$D' \in |D+H|$ is nonsingular, irreducible and intersects C' transversely.

Similarly, we can choose $E' \in |H|$ nonsingular, irred. and transverse to D', and $F' \in |H|$ nonsingular, irred.

transverse to C' and E'.

So we have $\quad C' \in |C+H|$, $D' \in |D+H|$

$$E' \in |H| \quad , F' \in |H|$$

all nonsingular, irreducible and transverse.

$$C \sim C' - E' \qquad D \sim D' - F'$$

$$C \cdot D = (C' - E') \cdot (D' - F')$$

$$= C' \cdot D' - C' \cdot F' - E' \cdot D' + E' \cdot F'$$

(A) $\quad C \cdot D = \# C' \cap D' - \# C' \cap F' - \# E' \cap D' + \# E' \cap F'$

which proves uniqueness.

To prove the existence, we take formula (A) as our definition for $C \cdot D$ and show that it is independent of the choices of the curves C', D', E', F' and the ample divisor H.

<u>Independence of the choice of C', D', E', F' for H fixed:</u>

For another choice of curves C'', D'', E'', F'' (similar to C', D', E', F') we need to show the intersection numbers are the same.

For instance, we show $\# C' \cap D' = \# C'' \cap D''$: this follows from the lemma via:

$$\# C' \cap D' = \deg \mathcal{O}_{C'}(D') = \deg \mathcal{O}_{C'}(D'') = C' \cdot D''$$

$$= \deg \mathcal{O}_{D''}(C') = \deg \mathcal{O}_{D''}(C'') = C'' \cdot D'' = \# C'' \cap D''.$$

Now we show independence of the choice of H:

For another very ample divisor H', choose curves

$C'' \in |C+H'|$, $D'' \in |D+H'|$, $E'' \in |H'|$, $F'' \in |H'|$

(after possibly replacing H' by a multiple)

nonsingular, irred., transverse.

We need to show:

$$\# C' \cap D' - \# C' \cap F' - \# E' \cap D' + \# E' \cap F'$$

$$= \# C'' \cap D'' - \# C'' \cap F'' - \# E'' \cap D'' + \# E'' \cap F''$$

We have $C \sim C' - E' \sim C'' - E''$

$$D \sim D' - F' \sim D'' - E''$$

via the Lemma, we obtain

$$C \cdot (D' - F') = C \cdot (D'' - F'') \text{ etc.}$$

exercise: finish the proof. $\square$.

<u>Def</u>: The self-intersection of any divisor is defined as

$D^2 := D \cdot D$. If D is a curve, i.e, an effective divisor, this is equal to $\deg \mathcal{O}_D(D)$.

<u>Example</u>: $X = \mathbb{P}^2$. $L \subset \mathbb{P}^2$ a line.

Then $L^2 = \deg \mathcal{O}_L(L) = \deg \mathcal{O}_L(1) = \deg \mathcal{O}_{\mathbb{P}^1}(1) = 1$

or $= \# L \cap L'$ $\quad L'$ a line $\neq L$.

If C is a curve of degree d in $\mathbb{P}^2$.

C is the scheme of zeros of $P \in H^0(\mathcal{O}_{\mathbb{P}^2}(d))$

$$C \sim dL \quad \text{because} \quad \mathcal{O}(dL) = \mathcal{O}(d) = \mathcal{O}(1)^{\otimes d}$$

So $\quad C^2 = (dL)^2 = d^2$

If D is a curve of degree e, then

$$C \cdot D = (dL) \cdot (eL) = de \quad \text{Bézout's theorem}$$

for curves in $\mathbb{P}^2$ when they are transverse.

<u>Proposition</u>: <u>The adjunction formula</u>:

For any nonsingular curve C on X of genus g, we have

$$2g - 2 = C \cdot (C + K_X)$$

Proof: We have the sequence of differentials on C:

$$0 \longrightarrow \mathcal{I}_C/\mathcal{I}_C^2 \longrightarrow \Omega^1_X|_C \longrightarrow \Omega^1_C \longrightarrow 0$$

$$\mathcal{I}_C/\mathcal{I}_C^2 = \mathcal{I}_C \otimes \mathcal{O}_X/\mathcal{I}_C = \mathcal{I}_C|_C = \mathcal{O}_X(-C)|_C$$
$$\cong \mathcal{O}_C(-C)$$

$$\Rightarrow \quad 0 \to \mathcal{O}_C(-C) \to \Omega^1_X|_C \longrightarrow \Omega^1_C \longrightarrow 0$$

Take top exterior powers:

$$\Lambda^2 \Omega^1_X|_C \cong \Omega^1_C \otimes \mathcal{O}_C(-C)$$
$$\omega_X|_C \cong \omega_C \otimes \mathcal{O}_C(-C)$$

$$\Rightarrow \quad K_X|_C \sim K_C - C|_C$$

or $$K_C \sim K_X|_C + C|_C$$

take degrees: $2g-2 = C\cdot(K_X + C)$ $\square$

<u>Theorem</u> (Riemann-Roch): For any divisor D

$$\chi(D) = \chi(\mathcal{O}_X) + \frac{1}{2} D\cdot(D - K_X)$$